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A review of some results of completely invariant components of meromorphic functions outside a small set

Una revisión de algunos resultados sobre componentes completamente invariantes de funciones meromorfas afuera de un conjunto pequeño

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Abstract

We consider the class $\mathcal{K}$ of functions f that are meromorphic outside a compact countable set $B(f)$, which is the closure of isolated essential singularities of f. We give a review of some results of functions in class $\mathcal{K}$ related to completely invariant components of the Fatou set. We state some open problems and give some examples which support them.

Fatou and Julia sets, Meromorphic functions

Resumen

Consideramos funciones en la clase $\mathcal{K}$ que son meromorfas afuera de un conjunto compacto contable $B(f)$, el cual es la clausura de singularidades esenciales aisladas de f. Presentamos algunos resultados de funciones en la clase $\mathcal{K}$ relacionados con componentes completamente invariantes del conjunto de Fatou. Enunciamos algunos problemas abiertos y proporcionamos algunos ejemplos que los ilustran.

Fatou y conjuntos de Julia, Funciones meromórficas

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Introduction

Between 1990 and 1991 I.N Baker, J. Kotus, and Y. Lü produced four papers [2, 3, 4, 5] in which they investigated the dynamics of transcendental meromorphic functions.

A. Bolsch [9] and M. Herring [14] in their PhD thesis investigated the iteration of analytic functions outside some set $B(f)$ of essential singularities, the set $B(f)$ is in some sense small. Their study was a natural generalization of the Fatou-Julia theory.

In particular, Bolsch in [8, 9, 10], investigated the iteration of analytic functions outside a compact countable set which is the closure of isolated essential singularities. In this class infinity may not be an essential singularity. If the set $B(f)$ has at least one essential singularity, removing the case when we have always omitted poles, then the study of the dynamics of this kind of functions is different from either rational functions or transcendental entire functions. We can think this class as the smallest semi group with the composition operation, containing all transcendental meromorphic functions. We denote this class by $\mathcal{K}$ following the notation in [12].

In this article we study the functions in class $\mathcal{K}$ and give a short review, with some examples, related to completely invariant components for $f \in \mathcal{K}$. In Section 2 we define the class $\mathcal{K}$, the Fatou and Julia sets and some of its properties. Section 3 contains generalizations of some results of transcendental meromorphic functions to functions in class $\mathcal{K}$. Also, we state some open problems and Conjectures with examples which support them.

The class $\mathcal{K}$ and the Fatou and Julia sets

We consider the $\mathcal{K}$ class of functions investigated by Bolsch in [8, 9, 10], which is the set of functions f from $\hat{\mathbb{C}}\backslash B(f)$ onto $\hat{\mathbb{C}}$ such that f is not constant and meromorphic in $\hat{\mathbb{C}}\backslash B(f)$, where the set $B(f)$ is a compact countable set and it is the closure of the set of isolated essential singularities of f.

We assume that $B(f)$ has at least one element e such that $f^{-1}(e) = p$, where p is called a *preimage under* f. Some examples of functions in class $\mathcal{K}$ are the following.

(i) $\lambda e^{R(z)} + \mu(z)$, where $R(z)$ and $\mu(z)$ are either rational functions or transcendental meromorphic functions.

(ii) $f_{\lambda,\mu}(z) = \tan(\lambda \tan(\mu z))$, for $\lambda, \mu \in \mathbb{C}$.

(iii) Composition of functions in (i) or (ii).

Bolsch in [9] proved the following result.

Proposition 2.1 *If f, g are in class $\mathcal{K}$, then the composition $f \circ g$ is in class $\mathcal{K}$, where*

$$B(f \circ g) = B(g) \cup g^{-1}\big(B(f)\big)$$

For $n \in \mathbb{N}$, we denote

$$f^{-n}(B(f)) = \{z : f^n(z) = e \in B(f)\}$$

The following theorem follows from [8] and [14].

Theorem 2.2 *If $f \in \mathcal{K}$, then $f^n \in \mathcal{K}$ and for each $n \geq 1$ the natural boundary of f^n is the set*

$$B_n = B(f^n) = \bigcup_{j=0}^{n-1} f^{-j}(B(f)),$$

so that f^n is meromorphic function in the region

$$D_n = \hat{\mathbb{C}}\backslash B_n.$$

Further, the sets B_n are all compact and countable. Moreover, f^n cannot be continued meromorphically over any point of B_n.

The *singular values* of the inverse function f^{-1}, $f \in \mathcal{K}$, consist of algebraic branch points or critical values together with the asymptotic values along paths which tend to $e \in B_n$, $n \in \mathbb{N}$. We denote

$$SV_n(f) = \{\text{singularities of } f^{-n},\quad n \in \mathbb{N}\},$$

where $SV_1(f) = \{\text{singularities of } f^{-1}\}$. By Theorem 7.12 in [14] we have:

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$$f^{n-1}(SV_1(f)\backslash B_{n-1}) \subset SV_n(f) \quad \subset \bigcup_{j=0}^{n-1} f^{-j}(SV_1(f)\backslash B_j),$$

with $B_j = \{z: f^j$ is not meromorphic at $z\}$ and $B_0 = \emptyset$.

The set

$SV = \{w\ \hat{\mathbb{C}}: f^{-n}$ has a singularity at w,for $n \in \mathbb{N}\}$ is given by

$SV = \cup_{j=0}^{\infty} f^{-j}(SV_1(f)\backslash B_j)$.

Observation. $\overline{SV} = J(f)$.

Definition 2.3 *Let* $f \in \mathcal{K}$.

(i) When $p \in f^{-n}(\infty)$, for some $n \geq 2$, p is called a *n- pre-pole of* f. For $n = 1$, p is called a *pole.*

(ii) When $e \in B(f)$ and $p \in f^{-n}(e)$, for some $n \geq 1$, p is called a *n-preimage under* f (or pre-singularity of f).

Following the notation in [12], we denote the set of all pre-singularities and essential singularities by

$$B^{-}(f) = \bigcup_{j=0}^{\infty} f^{-j}\big(B(f)\big).$$

For functions in class $\mathcal{K}$ we shall give the following definitions:

(i) w is an *omitted value* of $f \in \mathcal{K}$ if for every $z \in \hat{\mathbb{C}}\backslash B(f)$ the equation $f(z) - w = 0$ has no solutions.

(ii) w is a *Picard exceptional* value of $f \in \mathcal{K}$ if the equation $f(z) - w = 0$ has only finitely many solutions.

(iii) $E(f)$ is the set of *Fatou exceptional values* of f, that is, points whose inverse orbit $O^{-}(z)$ is finite, where $O^{-}(z) = \{w: f^n(w) = z$ for some $n \in N\}$.

Observations. (i) If w is an omitted value of $f \in \mathcal{K}$, then w is a Picard exceptional value of $f \in \mathcal{K}$.

(ii) When $B(f) = \{\infty\}$ is the only essential singularity and there exists a non-omitted pole, we are dealing with transcendental meromorphic functions.

If $f \in \mathcal{K}$ and $n \in \mathbb{N}$ is the minimum such that z satisfies $f^n(z) = z$, the point z is called a *periodic fixed point of period* n. When $n = 1$ the point z is called a fixed point.

If z is a fixed point of period n, the set $\{z_1 = z, z_i = f(z_{i-1}), z_{n+1} = z_1; 2 \leq i \leq n, n \in \mathbb{N}\}$ is called the cycle at z.

If z is a fixed point of period n, the *eigenvalue or multiplier* of the cycle at z is defined and denoted by

$$\lambda(z) = \prod_{j=1}^{n} f'(z_i).$$

When any value z_i is the point at infinity $f'(z_i)$ is replace by the derivative of $1/f(1/z)$ at the origin.

The classification of a periodic point z_0 of period n of $f \in \mathcal{K}$ is given as follows:

(a) if $|\lambda(z_0)| = 0$, z_0 is called *super-attracting*;

(b) if $0 < |\lambda(z_0)| < 1$, z_0 is called *attracting*;

(c) if $|\lambda(z_0)| > 1$, z_0 is called *repelling*;

(d) if $|\lambda(z_0)| = 1$ and $(f^n)'(z_0)$ is a root of unit, z_0 is called *rationally indifferent*, in this case z_0 is also known as a *parabolic periodic point*;

(e) if $|\lambda(z_0)| = 1$ and $\lambda(z_0)$ is not a root of unit, z_0 is called *irrationally indifferent*.

The definitions of the Fatou and Julia sets are as follows.

The *Fatou set*, denoted by $F(f)$, is the maximal open set U, such that all f^n are analytic and forms a normal family in U in the sense of Montel.

The *Julia set*, denoted by $J(f)$, is the complement of the Fatou set, that is,

$$J(f) = \hat{\mathbb{C}}\backslash F(f).$$

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Some basic properties of the Fatou and Julia set sare the folowing; see [9] for a proof.

(a) The Fatou set is open, so the Julia set is closed.

(b) The Julia set $J(f)$ is perfect.

(c) The set $F(f)$ is completely invariant, this means, it is forward invariant $f(F(f)) \subseteq F(f)$ and backward invariant $f^{-1}(F(f)) \subseteq F(f)$.

The set $J(f)$ is also completely invariant in the sense that $f(J(f)\backslash B(f)) \subseteq J(f)$ and $f^{-1}(J(f)) \subseteq J(f)$.

(d) For a positive integer p, $F(f^p) = F(f)$ and $J(f^p) = J(f)$.

(e) The Julia set $J(f)$ is the closure of the set of repelling periodic points of all periods of f.

(f) If U is an open set and $U \cap J(f) \neq \emptyset$, Then $\bigcap_n f^n(U)$ contains $\hat{\mathbb{C}}$ except for the omitted values.

For a function $f \in \mathcal{K}$ a Fatou component U can be either:

(a) *periodic* if $f^n(U) \subset U$, for some $n \geq 1$;

(b) *pre-periodic* if $f^m(U)$ is periodic for some integer $m \geq 0$ or

(c) *wandering* if U is neither periodic nor pre-periodic.

Consider $U_1 \subseteq F(f)$ a periodic component, we say that $\{U_i\}$, $1 \leq i \leq p$, is a *periodic cycle of period p* if $f(U_i) \subseteq U_{i+1}$, for all $1 \leq i \leq p-1$, and $f(U_p) \subseteq U_1$.

If U is a periodic component of the Fatou set of period n its classification is given similar to that for transcendental meromorphic functions, that is, U is either *an attracting component, a parabolic component, a Siegel disk, a Herman ring or a Baker domain*.

Invariant components of the Fatou set for $f \in \mathcal{K}$

In this section we will take $f \in \mathcal{K}$ and assume that $B(f)$ has at least one element with non empty pre-image, observe that with this assumption we are ruling out transcendental entire functions, since they cannot have poles and analytic functions in $\hat{\mathbb{C}}\backslash\{0,\infty\}$, while transcendental meromorphic functions are allowed. The following result is a corollary of Theorem 9.1.1 in [14] which applies to functions in class $\mathcal{K}$.

Theorem 3.1 *Let $f \in \mathcal{K}$. If U is a component of $F(f)$such that $B(f) \cap \partial U = \emptyset$, then $f\colon U \to f(U)$ is a finite branched cover.*

In holomorphic dynamics interesting problems are related to the connectivity of the Fatou components. We recall that a component U in the Fatou set is completely invariant if $f(U) \subset U$ and $f^{-1}(U) \subset U$. Concerning completely invariant components in the Fatou set Lemmas 4.1, 4.2 and 4.3 for transcendental meromorphic functions in [4] were generalized to a class of functions which contains the class $\mathcal{K}$ in [7] and [14]. The results for functions in class $\mathcal{K}$ are stated as follows.

Lemma 3.2 *Let $f \in \mathcal{K}$ and U be a completely invariant component of the Fatou set. Then $\partial U = J(f)$.*

Lemma 3.3 *Let $f \in \mathcal{K}$. If there are two or more completely invariant components in the Fatou set, then each one is simply connected.*

For a periodic component in the Fatou set, Bolsch in [9] improves Lemma 4.1 in [4] for invariant components.

Theorem 3.4 *If $f \in \mathcal{K}$ and U is a periodic component of the Fatou set, then U has connectivity 1, 2 or ∞.*

Lemmas 4.4 and 4.5 in [4] were generalized in [7] and [14] to a class of functions that contains the class $\mathcal{K}$, where the set of singular values SV is finite. We state the results for $f \in \mathcal{K}$.

Theorem 3.5 *Suppose that $f \in \mathcal{K}$, the set $SV_1(f)$ is finite and the Fatou set has a simply connected completely invariant component U. If $e \in B(f)$, then e is accessible in U.*

Theorem 3.6 *If $f \in \mathcal{K}$ and the set $SV_1(f)$ is finite, then the Fatou set has at most two completely invariant component.*

Examples of functions in class $\mathcal{K}$ that satisfies Theorems 3.5 and 3.6 are the following:

(i) $f_\lambda(z) = \lambda \tan z$, for some $\lambda \in \mathbb{C}$, see [15] for details.

(ii) $f_{\lambda,\mu}(z) = \tan(\lambda \tan(\mu z))$, with some conditions in $\lambda, \mu \in \mathbb{C}$.

For function sin class $\mathcal{K}$, where the set $SV_1(f)$ is not finite, we can state the well known open problem for transcendental meromorphic functions to functions in class $\mathcal{K}$.

Problem 1. *Is it true that for $f \in \mathcal{K}$, the Fatou set has at most two completely invariant components?*

We do not know examples of function f in class $\mathcal{K}$ with $SV_1(f)$ not finite that satisfies Problem 1, so we state the following problem.

Problem 2. Is there $f \in \mathcal{K}$, where the set SV_1 is not finite, such that the Fatou set has two completely invariant components in the Fatou set?.

One can ask: Is there $f \in \mathcal{K}$ with at most one completely invariant component? For transcendental meromorphic functions with finite many poles it was proved by Domínguez in [11], but the proof is very close to that in [1], for transcendental entire functions, which unfortunately has a missing case. Thus the proof of Theorem F in [11] is not correct . Therefore, we state the following conjecture.

Conjecture 1. *If f is a transcendental meromorphic function with at most finitely many poles, then there is at most one completely invariant component.*

There are examples which support Conjecture 1.

Example 1. A transcendental meromorphic function which satisfies hypothesis of Conjecture 1 is $f_\lambda(z) = \lambda \sin z + \frac{\epsilon}{z-\pi}$, where $0 < \lambda < 1$ and $\epsilon > 0$ is sufficiently small. Observe that f_λ has just one non-omitted pole π and the set $SV_1(f_\lambda)$ is countable infinite. The Fatou set of f_λ is a completely invariant component which is unbounded and multiply connected; see [11] for details.

Could Conjecture 1 be generalized to functions in class $\mathcal{K}$ with at least two essential singularities? We claim that we can substitute the poles for pre-images. Thus we state the following conjecture.

Conjecture 2. *If $f \in K$ with at most finitely many pre-images, i.e., $f^{-1}(e_i)$ is non empty with $e_i \in B(f)$, then there is at most one completely invariant component.*

The claim above is given by the following example.

Example 2. Take the composition

$h(f) = f_\lambda(f_\lambda(z))$ with $f_\lambda(z) = \lambda \sin z + \frac{\epsilon}{z-\pi}$, we obtain:

$$h_\lambda(z) = \lambda \sin(\lambda \sin z + \frac{\epsilon}{z-\pi}) + \frac{\epsilon}{(\lambda \sin z + \frac{\epsilon}{z-\pi}) - \pi},$$

where the pole π has become an essential singularity with non-empty preimages and $\epsilon > 0$ sufficiently small. It is not difficult to verify that for $0 < \lambda < 1$ the dynamics of f_λ and h_λ are similar. Thus, h_λ has a completely invariant component in the Fatou set.

Continuing with general results of completely invariant components in the Fatou set for $f \in \mathcal{K}$ the following theorem was proved in [6] for another classes of functions, but the proof applies to functions in class $\mathcal{K}$.

Theorem 3.7 *Let $f \in \mathcal{K}$ and suppose that that there is an attracting fixed point whose Fatou component U contains all the singular values of f. Then $J(F)$ is totally disconnected*

Examples of functions which satisfies hypothesis of Theorem 3.7 are the following:

(i) $f_\lambda(z) = \lambda \tan z$, for some $\lambda \in \mathbb{C}$, see [15] for details.

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(ii) $f_{\lambda,\mu}(z) = \lambda e^{\frac{1}{z^2+1}} + \mu$, for some $\lambda, \mu \in \mathbb{C}$, see [13] for details.

Observe that in Theorem 3.7 the Fatou set is just one completely invariant component which is multiply connected.

Another conjeture related to completely invariant components for functions in class $\mathcal{K}$ is the following.

Conjecture 3. *If* $f \in \mathcal{K}$, *the set* $SV_1(f)$ *is finite and the Fatou set is a completely invariant component, then every repelling periodic point is an accessible boundary point in the Fatou set.*

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