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Different types of homogeneity

Distintos tipos de homogeneidad

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Abstract

The homogeneity degree of a space X is the number of orbits under the action of the group of homeomorphisms of X in X. The smaller the homogeneity degree, the more homogeneous the space, in fact, having a homogeneity degree 1 is equivalent to being homogeneous. Another way to measure the homogeneity of a space is to use n-homogeneity and n-homogeneity at a point. Our work focuses on spaces called continua (metric, compact and connected spaces), studying the relationship between these three types of homogeneity mentioned, as well as their interaction with the local connectedness, the indecomposability and cut points of the continuum. We prove, among other things, that the n-homogeneous spaces at a point have homogeneity degree 1 or 2 and, when the space has exactly one cut point, we relate the homogeneity degree of the continuum with the homogeneity degree of the components of the complement of the cut point. In addition, we generalize some known results of the topic.

Continuous, Degree of homogeneity, n-homogeneity, Cut points

Resumen

El grado de homogeneidad de un espacio X es el número de órbitas bajo la acción del grupo de homeomorfismos de X en X. Entre más pequeño es el grado de homogeneidad, más homogéneo es el espacio, de hecho, tener grado de homogeneidad 1 es equivalente a ser homogéneo. Otra manera de medir la homogeneidad de un espacio es utilizar la n-homogeneidad y la n-homogeneidad en un punto. Nuestro trabajo se centra en los espacios llamados continuos (espacios métricos, compactos y conexos), estudiando la relación entre estos los tres tipos de homogeneidad mencionados, así como su interacción con la conexidad local, la indescomponibilidad y los puntos de corte del continuo. Probamos, entre otras cosas, que los espacios n-homogéneos en un punto tienen grado de homogeneidad 1 o 2 y, cuando el espacio tiene un único punto de corte, relacionamos el grado de homogeneidad del continuo con el grado de homogeneidad de las componentes del complemento de dicho punto de corte. Además, generalizamos algunos resultados ya conocidos del tema.

Continuo, Grado de homogeneidad, *n*-homogeneidad, Puntos de corte

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1.-Introduction

This work is framed within the Continuum Theory and more specifically, in the study of $\frac{1}{n}$-homogeneous continua. Research in homogeneous spaces is extensive in Topology. However, not all spaces are homogeneous and, even among them, it is interesting to have a measure of how much homogeneous is the space. We investigate three concepts that help us in this task: $\frac{1}{n}$-homogeneity, n-homogeneity and n-homogeneity at a point.

The study of $\frac{1}{n}$-homogeneous continua, for n>1, formally began in 1989 with H. Patkowska. However, it was not until few years ago that this topic has gained momentum, gaining the interest of researchers in Continuum Theory. Most of the papers written in recent years focus on the study of $\frac{1}{2}$-homogeneity.

In this work we study the three notions of homogeneity we mentioned, each one separately, and the relationship between them; as well as the interaction with the local connectedness, the indecomposability and the cut points of a continuum.

Section 2 is devoted to the definitions, the notation and the basic and general theorems that we will use. Section 3 is dedicated to $\frac{1}{n}$-homogeneous spaces. We define, among other notions, the homogeneity degree of a space, $\frac{1}{n}$-homogeneous space and orbits of a space. We also prove general results on this topic.

In section 4 we adapt, for n-homogeneous spaces at a point, theorems that holds for n-homogeneous spaces, and generalize other known results. In Theorems 4.4 and 4.7 we prove that the n-homogeneous or n-homogeneous spaces at a point have 1 or 2 as homogeneity degree, and are decomposable for $n \geq 2$. Theorems 4.8 to 4.10 relate the homogeneity degree of a space with the n-homogeneity and the n-homogeneity at a point. In section 5 we study the interaction of these concepts with the local connectedness.

In section 6 we present two easy results that show the relationship between the different types of homogeneity we have considered, when one of the orbits of the space is degenerate.

Finally, in section 7 we study $\frac{1}{n}$-homogeneous continua with exactly one cut point. Theorem 7.1 shows a relation between the orbits of the continuum and the orbits of the components of the complement of the cut point. Corollary 7.10 gives an example of a $\frac{1}{3}$-homogeneous continuum that is obtained by gluing copies of the same homogeneous and locally connected continuum.

2.-Preliminaries

We will denote by $\mathbb{N}$ to the set of natural numbers. If A is a set, its cardinality is denoted by $|A|$. We say that A is *degenerated* if $|A| = 1$ and, in any other case, A is *non degenerated*. Let X be a space and $A \subset X$. We will denote by $\mathbf{1}_X$ the identity map on X, by $Cl_X\,(A)$ and by $Int_X\,(A)$ the *closure* and the *interior* of A in X, respectively. If X is a metric space, $di\acute{a}m(A)$ is the diameter of A, and if f is a function wih domain X, by $f|_A$ we denote the restriction of f to the set A.

Throughout this work, all the spaces are T_1 and, in general, continua. A *continuum* is a metric, compact, connected and, for this paper, with more than one point. If X is a continuum, a *subcontinuum of* X is a closed, connected and non empty subset of X. We will denote by I the unit interval $[0, 1]$. Continua homeomorphic to I are called *arcs*. If A is an arc and h is a homeomorphism between I and A, the *end points* of A are $h(0)$ y $h(1)$.

Let X be a topological space. If $x \in X$, we say that X is locally connected at x, if for each open set U having x, there is an open and connected subset V of X that contains x and is contained in U. The space X is locally connected if X is locally connected at each one of its points. We say that the space X is *connected im kleinen* (or *cik)* at the point $x \in X$, if for every open set U containing x, there is a connected subset V of X having x in its interior and it is contained in U. Finally, X is arcwise connected if for each pair of points p and q in X, there is an arc whose ends are p and q.

Clearly, if X is locally connected at x, then X is connected im kleinen at x. The other implication is not always true, see for example Theorem 7.6 (Pacheco, 2009, p.69). However, it is known that a space is locally connected if and only if it is cik at all of its points (Theorem 7.8, Pacheco, 2009, p.73).

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Let X be a continuum that is not cik at some point x. From Theorem 5.12 (Nadler, 1992, p.76) we deduce that there exists a non-degenerate subcontinuum K of X that contains x and such that X is not cik at each point of K. So, if a continuum it is not cik at a point, then it is not cik at uncountably many points.

In Theorem 4.2 (Dugundji, 1966, p.123) locally connected spaces are characterized as those in which the components of the open sets are open sets. Throughout the present work, we will use these results in more than one occasion.

The proof of the following Theorem can be found in Theorem 3.3 (Dugundji, 1966, p.121).

Theorem 2.1
Let $f: X \to Y$ be a homeomorphism between metric spaces X and Y. If A is a component of a subset C of X, then $f(A)$ is a component of $f(C)$.

Theorem 2.2
Let X be a metric and compact space, A and B two non closed subsets of X, and c, d in X such that $Cl_X(A) = A \cup \{c\}$ and $Cl_X(B) = B \cup \{d\}$. If $f: A \to B$ is a homeomorphism, then $g: Cl_X(A) \to Cl_X(B)$ defined for each $x \in \mathrm{Cl}_X(A)$ by

$$g(x) = \begin{cases} f(x), & if\ x \in A; \\ d, & if\ x = c \end{cases}$$

is a homeomorphism extending f.

Proof
Since f is a bijective function, clearly g is also bijective. To see that g is continuous, we only need to prove that it is continuous at c. For this, suppose that $\{a_n\}_n$ is a sequence of points in $Cl_X(A)$ converging to c and $\{g(a_n)\}_n$ converges to a point $b \in Cl_X(B)$. We want to see that $b = d$. Suppose that $b \in B$ and put $a = g^{-1}(b) = f^{-1}(b)$. Note that $a \in A$. Without loss of generality suppose that each $a_n \in A$, this implies that $g(a_n) = f(a_n)$. Since f^{-1} is a homeomorphism from B to A and $\{f(a_n)\}_n$ is a sequence in B whose limit is also in B, from Proposition 6.1.5 (Margalef, 1979, p.6) we obtain:

$$a = f^{-1}\left(\lim_{n\to\infty} f(a_n)\right) = \lim_{n\to\infty} f^{-1}(f(a_n)) = \lim_{n\to\infty} a_n = c$$

This implies that $c = a$, and since $a \in A$, c is an element of A. Thus, A is closed in X. From this contradiction, we conclude that $b = d = g(c)$. Hence g is continuous at c. As g is a function from a compact space to a metric space, it is closed and thus, it is a homeomorphism. □

The first part of the following result is known as the Boundary Bumping Theorem. Its proof can be found in Theorem 5.6 of (Nadler, 1992, p.74). The second part is a consequence of the first, and its proof can be found in Corollary 5.9 (Nadler, 1992, p.75).

Theorem 2.3
Let X be a continuum and E a non empty proper subset of X. Then the following statements hold.

1. If A is a component of E; then $\mathrm{Cl}_X(A) \cap \mathrm{Cl}_X(X - E) \neq \emptyset$.
2. If E is a subcontinuum of X and A is a component of $X - E$, then $A \cup E$ is a subcontinuum of X.

Let X be a connected space. We say that $c \in X$ is a *cut point* of X if $X - \{c\}$ is not connected. The symbol $Cut(X)$ will denote the set of cut points of X (note that $Cut(X)$ could be empty). If $c \in X$, by $\mathcal{A}_c$ we denote the family of the components of $X - \{c\}$.

The following result shows properties of the components of the complement of a cut point in a topological space. Recall that two non-empty sets H and K of a space X are mutually separated if the closure of H does not intersect K and the closure of K does not intersect H. It is known that a space X is connected if and only if there are not two mutually separated sets whose union is X.

Theorem 2.4
Let X be a continuum and $c \in X$. For each component A of $X - \{c\}$ the following properties hold.

1. $Cl_{X(A)} = A \cup \{c\}$ and $Cut\big(Cl_X(A)\big) \subset A$.
2. If A is open in X, then every cut point of $Cl_X(A)$ is a cut point of X.
3. If $Cut(X) = \{c\}$ and A is open in X, then $Cl_X(A)$ is a subcontinuum of X without cut points (of itself).

Proof
Let A be a component of $X - \{c\}$. Since $X - \{c\}$ is a proper, non-empty subset of X; by part 1 of Theorem 2.3 (applied to $E = X - \{c\}$), it happens that

$$\emptyset \neq Cl_X(A) \cap Cl_X(X-(X-\{c\}) = Cl_X(A) \cap \{c\}.$$

Then $c \in Cl_X(A)$ and $A \cup \{c\} \subset Cl_X(A)$. From part 2 of Theorem 2.3 (applied to $E = \{c\}$), we obtain $A \cup \{c\}$ is a subcontinuum of X. This proves that $Cl_X(A) = A \cup \{c\}$. Now suppose that a is a cut point of $Cl_X(A)$. As $Cl_X(A) - \{c\} = A$ and A is connected, we have $a \neq c$. Therefore, $a \in Cl_X - \{c\} = A$. This proves 1. Now suppose that A is open in X and a is a cut point of $Cl_X(A)$. From 1, we obtain $a \in A$. Let's define

$$B = \bigcup\{C \in \mathcal{A}_c : C \neq A\}.$$

From 1, for each $C \in \mathcal{A}_c - \{A\}$, we have $c \in Cl_X(C) \subset Cl_X(B)$. Since A is open, $X - A = B \cup \{c\}$ is a closed subset of X that contains B. Then, $Cl_X(B) = B \cup \{c\}$ and, since $Cl_X(A) = A \cup \{c\}$ and $A \cap B = \emptyset$, we obtain $A \cap Cl_X(B) = \emptyset$ and $Cl_X(A) \cap B = \emptyset$. Now, since a is a cut point of $Cl_X(A)$, there are two non empty and mutually separated sets H and K such that $Cl_X(A) - \{a\} = H \cup K$. Suppose without loss of generality that $c \in K$. Then $H \subset A$ and

$$X - \{a\} = (Cl_X(A) - \{a\}) \cup B = (H \cup K) \cup B = H \cup (K \cup B).$$

In addition, H and $K \cup B$ are non-empty subsets of X such that

$$Cl_X(H) \cap (K \cup B) = (Cl_X(H) \cap K) \cup (Cl_X(H) \cap B) = Cl_X(H) \cap B \subset Cl_X(A) \cap B = \emptyset$$

And

$$H \cap Cl_X(K \cup B) = (H \cap Cl_X(K)) \cup (H \cap Cl_X(B) = H \cap Cl_X(B) \subset A \cap Cl_X(B) = \emptyset.$$

Hence a is a cut point of X, which proves 2. To see 3, suppose that $Cut(X) = \{c\}$, A is a component of $X - \{c\}$ and that A is open in X. Then $Cl_X(A)$ is a subcontinuum of X, and from 1 and 2, $Cut(Cl_X(A)) \subset A \cap Cut(X) = A \cap \{c\} = \emptyset$. Thus, $Cut(Cl_X(A)) = \emptyset$.□

3. 1/n-homogeneous spaces

Given a space X, we denote by $\mathcal{H}(X)$ the group of homeomorphisms of X in X. The orbit $\mathrm{Orb}(x)$ of x in X is the orbit of x under the action of $\mathcal{H}(X)$ on X, i.e.

$$Orb_X(x) = \{h(x) : h \in \mathcal{H}(X)\}.$$

The family of orbits of a space X is a partition of X and a set $\mathcal{O}$ is an orbit of X if and only if there is $x \in X$ such that $\mathcal{O} = Orb_X(x)$.

The homogeneity degree is the number of orbits of X. Alternatively, we say that X is $\frac{1}{n}$-homogeneous to mean that X has homogeneity degree n. We are interested in spaces with a finite homogeneity degree. The following result provides some additional properties of the orbits of a space.

Theorem 3.1
Let X be a space and $\mathcal{O}$ an orbit of X. The following statements hold.

1. If $f \in \mathcal{H}(X)$, then $f(\mathcal{O}) = \mathcal{O}$.
2. If $\mathcal{O}_1$ is an orbit of X such that $\mathcal{O} \cap \mathrm{Cl}_X(\mathcal{O}_1) \neq \emptyset$, then $\mathcal{O} \subset \mathrm{Cl}_X(\mathcal{O}_1)$.
3. If $\mathrm{Int}_X(\mathcal{O}) \neq \emptyset$, then $\mathcal{O}$ is open in X.

Proof
First, take $z \in X$ such that $\mathcal{O} = Orb_X(z)$. Take $f \in \mathcal{H}(X)$ and $x \in \mathcal{O}$. There exists $h \in \mathcal{H}(X)$ such that $h(z) = x$. Since $(f \circ h)(z) = f(h(z)) = f(x)$, then $f(x) \in Orb_X(z) = \mathcal{O}$. Then, $f(\mathcal{O}) \subset \mathcal{O}$ and, by a similar argument, $f^{-1}(\mathcal{O}) \subset \mathcal{O}$. Therefore, $\mathcal{O} = f(\mathcal{O})$.

To see 2 let $x \in \mathcal{O}$ and U be an open set in X having x. Take $y \in \mathcal{O} \cap Cl_X(\mathcal{O}_1)$. There exists $f \in \mathcal{H}(X)$ such that $f(x) = y$. Since $f(U)$ is an open set in X such that $y \in f(U)$ and $y \in Cl_X(\mathcal{O}_1)$, it follows that $f(U) \cap \mathcal{O}_1 \neq \emptyset$. From here, applying part 1 to the homeomorphism f^{-1} we obtain

$$\emptyset \neq f^{-1}(f(U) \cap \mathcal{O}_1) = U \cap \mathcal{O}_1.$$

Hence $x \in Cl_X(\mathcal{O}_1)$, which proves 2. To see 3, take $y \in \mathcal{O}$ and $x \in Int_X(\mathcal{O})$ and $f \in \mathcal{H}(X)$ such that $f(x) = y$. Then $f(Int_X(\mathcal{O}))$ is open in X and it contains y. That $f(Int_X(\mathcal{O})) \subset f(\mathcal{O}) = \mathcal{O}$ follows from 1. Therefore, $\mathcal{O}$ is open in X. This proves 3. The following theorem generalizes part 1 of Theorem 3.1 and its proof is similar to it.

Theorem 3.2
Let X and Y be two spaces and $f: X \to Y$ a homeomorphism. If $z \in X$, then

$$f(Orb_X(z)) = Orb_Y(f(z)).$$

Corollary 3.3
Homeomorphic spaces have the same homogeneity degree.

Theorem 3.4
Let X be a locally connected continuum with exactly one cut point c. If A and B are two homeomorphic elements of $\mathcal{A}_c$, then there is a homeomorphism $F: X \to X$ such that $F(A) = B$, $F(c) = c$ and, for each $a \in A$

$$F\big(Orb_A(a)\big) = Orb_B\big(F(a)\big)$$

Proof
Fix $A \in \mathcal{A}_c$. Let $B \in \mathcal{A}_c$ be homeomorphic to A. By the first part of Theorem 2.4, $Cl_X(A) = A \cup \{c\}$ and $Cl_X(B) = B \cup \{c\}$. Since A and B are homeomorphic, by Theorem 2.2 we can choose a homeomorphism $g: A \cup \{c\} \to B \cup \{c\}$ such that $g(A) = B$ and $g(c) = c$ (if $B = A$ we take g as the identity). Since X is locally connected and, A and B are components of $X - \{c\}$, A and B are open in X. Hence $X - (A \cup B)$ is closed in X. Define $F: X \to X$ as follows:

$$F(X) = \begin{cases} g(x), \ if\ x \in Cl_X(A); \\ g^{-1}(x), \ if\ x \in Cl_X(B); \\ x, \ \ if\ x \in X - (A \cup B). \end{cases}$$

Clearly, F is a homeomorphism and $F(A) = B$. Furthermore, if $a \in A$, as $F|_A$ is a homeomorphism from A to B, by Theorem 3.2, $F\,(Orb_A(a)) = Orb_B(F(a))$. This concludes our proof. □

4. n-homogeneous spaces

Definition 4.1
Let X be a space and $n \in \mathbb{N}$. We say that X is n-homogeneous if, for each pair of sets A and B, each one with exactly n elements, there exists $h \in \mathcal{H}(X)$ such that $h(A) = B$. We say that X is n-homogeneous at a point $c \in X$ if, for each pair of sets A and B having c, with exactly n elements, there exists $h \in \mathcal{H}(X)$ such that $h(A) = B$ and $h\,(c) = c$.

As a consequence of the following theorem, n-homogeneity is a topological invariant.

Proposition 4.2
Let f be a homeomorphism between X and Y, and $n \in \mathbb{N}$. If X is n-homogeneous at p, then Y is n-homogeneous at $f\,(p)$.

Proof
Let A and B be two subsets of Y with exactly n elements and such that $f(p) \in A \cap B$. Note that $f^{-1}(A)$ and $f^{-1}(B)$ are two subsets of X having p, each one with exactly n elements. Since X is n-homogeneous at p, there is a homeomorphism $h: X \to X$ such that $h\,(f^{-1}\,(A)) = f^{-1}\,(B)$ and $h\,(p) = p$. Hence $f \circ h \circ f^{-1}: Y \to Y$ is a homeomorphism, which sends A in B and fixes $f\,(p)$. This proves that Y is n-homogeneous at $f\,(p)$. Note that the definition of 1-homogeneous space coincides with the definition of $\frac{1}{1}$-homogeneous space, these spaces are simply called homogeneous. Note that all spaces are trivially 1-homogeneous at each one of their points. For this reason, for the study of the n-homogeneous continuums at a point, we will always consider $n \geq 2$. The first part of the following theorem is proved in Theorem 1 (Burges, 1954, p.137), while the second is shown in Corollary 2 (Brown, 1959, p.647).

Theorem 4.3
Let X be a space and $n \in \mathbb{N}$. If X is n-homogeneous, then the following statements hold.

1. X is homogeneous.
2. If $n \geq 2$, X is $(n - 1)$ -homogeneous.

We present the corresponding version of Theorem 4.3, but for n-homogeneous spaces at a point.

Theorem 4.4
Let X be a continuum and $n \in \mathbb{N}$. If X is n-homogeneous at a point $p \in X$, then the following statements hold.

1. X is homogeneous or X is $\frac{1}{2}$-homogeneous and its orbits are $X - \{p\}$ and $\{p\}$.
2. If $n \geq 2$, X is $(n - 1)$ -homogeneous at p.

Proof
Suppose that X is n-homogeneous at the point $p \in X$. Let us first show that $X - \{p\}$ is contained in an orbit of X. Take two points x and y in $X - \{p\}$ and consider $n - 2$ points $x_1, x_2, \dots, x_{n-2}$ in $X - \{x, y, p\}$. Since X is n-homogeneous at p, there exists $h \in \mathcal{H}\,(X)$ such that:

$h(\{x, x_1, x_2, \dots, x_{n-2}\}) = \{y, x_1, x_2, \dots, x_{n-2}\}$ and $h(p) = p$.

If $h(x) = y$, then x and y belong to the same orbit of X. Suppose that $h(x) \neq y$. Consequently, there exists $i_1 \in \{1,2,\dots,n-2\}$ such that $h(x) = x_{i_1}$. If $h(x_{i_1}) = y$, then $h^2 \in \mathcal{H}(X)$ and $h^2(x) = y$. We continue in this fashion to obtain $m \in \{1,2,\dots,n-1\}$ such that $h^m(x) = y$, which proves that x and y belong to the same orbit of X. Therefore, $X - \{p\}$ is contained in an orbit $\mathcal{O}$ of X.

If X is not homogeneous, p does not belong to $\mathcal{O}$, and the two orbits of X are $X - \{p\}$ and $\{p\}$. This proves 1.

To prove 2 note that the case $n = 2$ is trivial. Then, we can assume that $n \geq 3$. From Definition 4.1, it follows that $X - \{p\}$ is $(n-1)$-homogeneous. Furthermore, by the second part of Theorem 4.3, we have $X - \{p\}$ is also $(n-2)$-homogeneous. Let A and B be two subsets of X, each one with exactly $n-1$ elements, and such that $p \in A \cap B$. There exists a homeomorphism $h: X - \{p\} \to X - \{p\}$ such that $h(A - \{p\}) = B - \{p\}$. Now, using Theorem 2.2, we can extend h to a homeomorphism $g: X \to X$, with $g(p) = p$. Consequently, $g(A) = B$ and $g(p) = p$. Therefore, X is $(n-1)$-homogeneous at point p. □

Definition 4.5
Let X be a continuum and $p \in X$. The composant of p in X is the union of the proper subcontinua of X having p. We say that X is decomposable if there are two subcontinua A and B of X such that $X = A \cup B$. Finally, we say that X is indecomposable if X is not decomposable.

The first part of the following theorem is proved in Theorem 2 (Kuratowski, 1968, p.209). The second part can be found in Theorem 11.15 (Nadler, 1992, p.203) and Theorem 11.17 (Nadler, 1992, p.204).

Theorem 4.6
For a continuum X, the following statements hold.

1. Each composant of X is dense in X.
2. If X is indecomposable, X has uncountable many composants and each two of them are disjoint.

Theorem 4.7
Let X be a continuum and $m \geq 2$. If X is m-homogeneous or m-homogeneous at some point, then X is decomposable.

Proof
Suppose that X is indecomposable and m-homogeneous at a point $c \in X$. From the second part of Theorem 4.4, it follows that X is 2-homogeneous at c. Let K_c be the composant of X containing c. Since composants are dense (Theorem 4.6), we can take $a \in K_c - \{c\}$. If h is a homeomorphism from X onto itself, which fixes c, then $h(K_c) = K_c$. Hence $h(\{a, c\}) \subset K_c$. Now, if we take b in another composant of X, it is not possible to find a homeomorphism that fixes c and sends a onto b, which contradicts that X is m-homogeneous at c.

Now we will assume that X is m-homogeneous and indecomposable. By the second part of Theorem 4.3, X is 2-homogeneous. Let K be a composant of X, $a, b \in K$ and $c \in X - K$. Since X is 2-homogeneous, there is a homeomorphism $h: X \to X$ such that $h(\{a,b\}) = \{a,c\}$. This gives $h(K) \cap K \neq \emptyset$, which implies $h(K) = K$ but, on the other hand, it also indicates that $c \in h(K) = K$. Since this contradicts the choice of c, we conclude that X is decomposable. □

As we indicated in part 1 of Theorem 4.3, if a space is n-homogeneous, it is homogeneous. However, given a homogeneous space Y, there does not always exist $n \in \mathbb{N}$ such that Y is n-homogeneous. To see this, by Theorem 4.7, it is enough to consider a homogeneous and indecomposable continuum, for example, the pseudoarc. The following result generalizes Lemma 4.4 of (Nadler, 2007, p.2158).

Theorem 4.8
Let X be a space and $n \in \mathbb{N}$. If X is n-homogeneous at two different points, then X is homogeneous.

Proof
Suppose that a and b are two points in X such that X is n-homogeneous at each one of them and that X is not homogeneous. Since X is n-homogeneous at a, by Theorem 4.4, X is $\frac{1}{2}$-homogeneous and its two orbits are $X - \{a\}$ and $\{a\}$. Similarly, since X is also n-homogeneous at b, then $X - \{b\}$ and $\{b\}$ are the two orbits of X. This is a contradiction, because $a \neq b$. Therefore, X is homogeneous.

Theorem 4.9
Let X be a homogeneous space and $n \in N$. If X is n-homogeneous at some point, then X is n-homogeneous.

Proof
Suppose that X is n-homogeneous at a point $p \in X$. Since X is homogeneous, by Proposition 4.2, X is n-homogeneous at each one of its points. Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be two subsets of X with exactly n points and h, $g \in \mathcal{H}(X)$ such that

$$h(\{x_1, \ldots, x_{n-1}, x_n\}) = \{y_1, \ldots, y_{n-1}, x_n\},$$
$$h(x_n) = x_n,$$
$$g(\{y_1, y_2, \ldots, y_{n-1}, x_n\}) = \{y_1, y_2, \ldots, y_n\}$$
$$\text{and} \quad g(y_1) = y_1.$$

Then, $(g \circ h)(\{x_1, \ldots, x_n\}) = \{y_1, y_2, \ldots y_n\}$. Therefore, X is n-homogeneous.

From Theorems 4.8 and 4.9 we obtain the following result.

Theorem 4.10
Let X be a space and $n \in \mathbb{N}$. If there are two different points in X such that X is n-homogeneous at each of them, then X is n-homogeneous.

5. Relationship with local connectedness

We have proved that m-homogeneous continua, with $m \geq 2$, are decomposable (Theorem 4.7). It turns out that, for each $m \geq 2$, m-homogeneous continua are locally connected (Corollary 5.2). However, not all m-homogeneous spaces are locally connected. Also, if a continuum is m-homogeneous at a point, not always is locally connected (Example 5.3).

In (Theorem 3.12, Ungar, 1975) the following result is shown.

Theorem 5.1
If X is a 2-homogeneous continuum, X is locally connected.

Corollary 5.2
Let X be a continuum and $m \geq 2$. The following statements are satisfied.

1. If X is m-homogeneous, X is locally connected,
2. If X is m-homogeneous at $p \in X$, then X is locally connected if and only if X is cik at some point in $X - \{p\}$.

Proof
Part 1 is obtained from Theorems 4.3 and 5.1. For the second part, first suppose that X is cik at some point of $X - \{p\}$. Since $X - \{p\}$ is contained in an orbit of X (Theorem 4.4), then X is cik at each point of $X - \{p\}$. Since the set of points at which X is not cik is infinite (Corollary 5.13, Nadler, 1992, p.78), X is also cik at p. Therefore, X is cik in each of its points and thus, X is locally connected. The other implication is clear.

Example 5.3
There are m-homogeneous spaces, m-homogeneous spaces at some point, even m-homogeneous continua at some point that are not locally connected.

Proof
It is known that, for each $m \in \mathbb{N}$, the Cantor set C is m-homogeneous (it is not difficult to see this if we consider that C is homeomorphic to $\{0,1\}^{\omega}$) and C is not locally connected.

Put $Y = \{0\} \cup \{1 / n : n \in \mathbb{N}\}$. It is easy to see that, for each $m \in \mathbb{N}$, Y is m-homogeneous at 0 and not locally connected at 0. Now, take the circumference S^1 and a fixed point $s \in S^1$. Let C be the usual Cantor set, X be the continuum $(S^1 \times C) / (\{s\} \times C)$ and $q: S^1 \times C \to X$ be the quotient function (See Figure 1). Note that $q(\{s\} \times C) = p$, for some $p \in X$, and $\{p\}$ is an orbit of X. In addition, $X - \{p\}$ is homogeneous and has no points of local connectedness. Let's see that X is 2-homogeneous at p. To this end take $a, b \in X - \{p\}$. Since $X - \{p\}$ is homogeneous, there exists a homeomorphism f of $X - \{p\}$ onto itself such that $f(a) = b$. By Theorem 2.2, we can extend f to a homeomorphism $g \in \mathcal{H}(X)$ such that $g(p) = p$. Since $a \in X - \{p\}$, we have $g(a) = f(a) = b$. Therefore, X is a continuum which is 2-homogeneous a p. It is easy to see that X is not locally connected at any point of $X - \{p\}$.

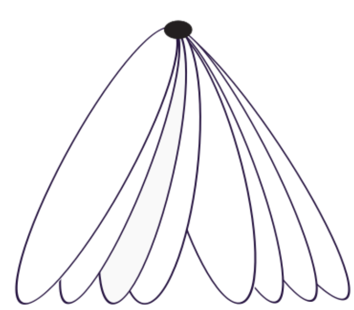

Figure 1 Continuum X of Example 5.3. Own elaboration

6. n-homogeneity and cut points

In this section we show a relationship between the different types of homogeneity we have considered, when one of the orbits is degenerate. The following proposition generalizes Lemma 4.3 of (Nadler, 2007, p.2188).

Proposition 6.1
Let X be a continuum with a degenerate orbit $\{c\}$. The following statements are equivalent.

1. X is $\frac{1}{2}$-homogeneous.
2. There is $m \geq 2$ such that X is m-homogeneous at c.
3. X is 2-homogeneous at c.

Proof
Suppose that X is $\frac{1}{2}$-homogeneous. The two orbits of X are $\{c\}$ and $X - \{c\}$. Consequently, if x and y are two points on $X - \{c\}$, there exists $h \in \mathcal{H}(X)$ such that $h\,(x) = y$ and since c is the only element of its orbit, $h\,(c) = c$. Then, X is 2-homogeneous at c. This proves that 1 implies 3.
It is clear that 3 implies 2.
Now suppose that there is $m \geq 2$ such that X is m-homogeneous at c. Since X is not homogeneous, by part 1 of Theorem 4.4, X is $\frac{1}{2}$-homogeneous. This completes the proof. □

Theorem 6.2
Let X be a continuum such that $Cut\,(X) \neq \emptyset$. Then the following statements hold.
1. If $m \in \mathbb{N}$, then X is not m-homogeneous;
2. If X is 2-homogeneous at some point c, then $Cut\,(X) = \{c\}$,
3. If $m \in \mathbb{N} - \{1,2\}$, then X is not m-homogeneous at any of its points.

Proof
By Theorem 6.6 (Nadler, 1992, p.89), X has at least two non-cut points. Since $Cut\,(X) \neq \emptyset$, X is not homogeneous. From part 1 of Theorem 4.3, we obtain 1. To see 2, suppose that X is 2-homogeneous at a point c. By the first part of Theorem 4.4, its two orbits are $X - \{c\}$ and $\{c\}$. On the other hand, $Cut\,(X)$ and $X - Cut\,(X)$, are also the two orbits of X. Since $X - Cut\,(X)$ is not degenerate, $Cut\,(X) = \{c\}$. This proves 2.

To see 3, suppose that X is m-homogeneous at a point $c \in X$, for some $m \geq 3$. From part 2 of Theorem 4.4, X is 2-homogeneous at c and, by part 2 of this theorem, $Cut\,(X) = \{c\}$. Lets take a component C of $X - \{c\}$, and $A \subset C$, with exactly $m - 1$ points. Since c is a cut point of X, then $C \neq X - \{c\}$. Being $m \geq 3$, we can take $B \subset X - \{c\}$, with exactly $m - 1$ elements, such that $B \cap C \neq \emptyset \neq B \cap (X - C)$. As the image of a component of $X - \{c\}$, under a homeomorphism of X in X, is a component of $X - \{c\}$, for each $h \in \mathcal{H}\,(X)$, $h\,(A) \neq B$. Therefore, X is not m-homogeneous at c.

7. $\frac{1}{n}$ − homogeneity and cut points

Let X be a connected space and $c \in X$. Recall that by $\mathcal{A}_c$ we mean the family of all the components of $X - \{c\}$. In this section we show some results that relate the orbits of a continuum with exactly one cut point c, with the orbits of the components of $X - \{c\}$.

The following theorem shows that if the continuum X is locally connected, then each orbit of X intersects, at most, one orbit of each component and, in addition, this orbit can be obtained as the union of the orbits of the components which it intersects.

Theorem 7.1
Let X be a continuum with a single cut point c. If $A \in \mathcal{A}_c$ and $a \in A$, the following statements are satisfied.

1. $\mathrm{Orb}_X(a) \cap A \subset \mathrm{Orb}_A(a)$.
2. $\mathrm{Orb}_X(a) \subset \cup \{\mathrm{Orb}_B(b) : B \in \mathcal{A}_c \text{ y } b \in \mathrm{Orb}_X(a) \cap B\}$.

Moreover, if X is locally connected, equalities are obtained in 1 and 2.

Proof
To prove 1, let $y \in Orb_X\,(a) \cap A$ and $h \in \mathcal{H}\,(X)$ such that $h\,(a) = y$. Since a and $h\,(a)$ are elements of A, by Theorem 2.1, $h\,(A) = A$. Hence $h|_A : A \to A$ is a homeomorphism and $y \in Orb_A(a)$.

To see 2, take $b \in Orb_X\,(a)$. Since c is the only cut point of X, a and c do not belong to the same orbit of X. Hence, $b \neq c$ and exists $B \in A_c$ such that $b \in B$. Therefore, $b \in Orb_X(a) \cap B$. Since $b \in Orb_B\,(b)$, this proves 2.

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Now suppose that X is locally connected and let us prove that the equalities are obtained in 1 and 2. First, take a point $y \in Orb_A(a)$. Then there is a homeomorphism $f: A \to A$ such that $f(a) = y$. By part 1 of Theorem 2.4, we have $Cl_X(A) = A \cup \{c\}$. Furthermore, by Theorem 2.2, we can extend f to a homeomorphism g, of $Cl_X(A)$ onto itself, such that $g(c) = c$. Now consider the function $h: X \to X$ defined, for each $x \in X$, as

$$h(x) = \begin{cases} g(x), & if\ x \in Cl_X(A); \\ x, & if\ x \in X - A. \end{cases}$$

Since X is locally connected, $X - A$ is closed in X. In addition,

$$Cl_X(A) \cap (X - A) = (A \cup \{c\}) \cap (X - A) = \{c\}.$$

Then, h is continuous, $h(c) = c$ y, since g and 1_{X-A} are homeomorphisms, h is a homeomorphism. As $h \in \mathcal{H}(X)$ and $h(a) = g(a) = f(a) = y$, $y \in Orb_X(a) \cap A$. This proves that $Orb_A(a) \subset Orb_X(a) \cap A$. Consequently

$$Orb_A(a) = Orb_X(a) \cap A, \tag{7.1}$$

To see the equality in the second part, take $y \in \cup\{Orb_B(b): B \in \mathcal{A}_c$ and $b \in Orb_X(a) \cap B\}$. Then there are $B \in \mathcal{A}_c$ and $b \in Orb_X(a) \cap B$ such that $y \in Orb_B(b)$. From (7.1), we have $Orb_B(b) = Orb_X(b) \cap B$, which gives $y \in Orb_X(b)$. Since $b \in Orb_X(a)$, we obtain $Orb_X(a) = Orb_X(b)$. This proves that $y \in Orb_X(a)$. □

Lemma 7.2

Let X be a locally connected continuum with exactly one cut point c. If every two elements of $\mathcal{A}_c$ are homeomorphic, for each $x \in X - \{c\}$ and each $B \in \mathcal{A}_c$, we have $Orb_X(x) \cap B \neq \emptyset$.

Proof

Suppose that all the elements of $\mathcal{A}_c$ are homeomorphic to each other, so they have the same degree of homogeneity. Fix $A \in \mathcal{A}_c$ and let $B \in \mathcal{A}_c$ and $a \in A$. As we proved in Theorem 3.4, there is $h_B \in \mathcal{H}(X)$ such that $h_B(c) = c$, $h_B(A) = B$ and $h_B(Orb_A(a)) = Orb_B(h_B(a))$.

If $B = A$, h_B is the identity map and, from part 1 of Theorem 7.1,

$$Orb_X(a) \cap A = Orb_A(a). \tag{7.2}$$

Suppose then that $B \neq A$. Let's prove that

$$Orb_X(a) \cap B = Orb_B(h_B(a)). \tag{7.3}$$

Let $x \in Orb_X(a) \cap B$ and $g \in H(X)$ such that $g(x) = a$. Then $h_B \circ g: X \to X$ is a homeomorphism such that $(h_B \circ g)(x) = h_B(a)$. Thus, $x \in Orb_X(h_B(a))$ and, from 1 of Theorem 7.1, we obtain $x \in Orb_B(h_B(a))$. This proves that $Orb_X(a) \cap B \subset OrbB\,(h_B(a))$. Now take $x \in Orb_B(h_B(a))$. Note that $x \in B$ and, again by 1 of Theorem 7.1, $x \in Orb_X(h_B(a))$. Let $g \in \mathcal{H}(X)$ such that $g(x) = h_B(a)$, then $h_B^{-1} \circ g \in \mathcal{H}(X)$ and

$$(h_B^{-1} \circ g)(x) = h_B^{-1}(g(x)) = h_B^{-1}(h_B(a)) = a$$

Therefore, $x \in Orb_X(a) \cap B$. This proves that $Orb_B(h_B(a)) \subset Orb_X(a) \cap B$. From which we obtain (7.3). By (7.2) and (7.3) it follows that the orbit of a in A intersects each element of $\mathcal{A}_c$. To finish the proof, take $x \in X - \{c\}$ and $B \in \mathcal{A}_c$. Suppose without loss of generality that $x \in A$. Applying (7.2) and (7.3) to $a = x$, we have $Orb_X(x) \cap B = Orb_B(h_B(x)) \neq \emptyset$.□

Lemma 7.3

Let X be a $\frac{1}{n}$-homogeneous continuum such that $Cut\,(X) = \{c\}$, $A \in \mathcal{A}_c$ and n_A be the number of orbits of X that intersect A. Then $n_A < n$, the homogeneity degree of A is less than or equal to n_A, and it is equal when X is locally connected.

Proof

Let χ_A be the family of the orbits of X that intersect A. Since $\{c\} \notin \chi_A$ then $n_A < n$. From part 1 of Theorem 7.1, each element of χ_A is contained in an orbit of A, which implies the homogeneity degree of A is at most $|\chi_A| = n_A$.

Now if X is locally connected, by Theorem 7.1, each element of χ_A is an orbit of A, which implies that the homogeneity degree of A is n_A. □

Theorem 7.4

Let X, c and $\mathcal{A}_c$ as in Lemma 7.2 and $n \geq 1$. Then X is $\frac{1}{n+1}$-homogeneous if and only if each $B \in \mathcal{A}_c$ is $\frac{1}{n}$-homogeneous.

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Proof

First suppose that each $B \in \mathcal{A}_c$ is $\frac{1}{n}$-homogeneous and their orbits are $Orb_B(a_{1_B}), Orb_B(a_{2_B}), \ldots, Orb_B(a_{n_B})$. Fix $A \in \mathcal{A}_c$. For each B, let $h_B \in H(X)$ such that $h_B(c) = c$, $h_B(A) = B$ and the image of an orbit $Orb_A(a)$ under h_B is $Orb_B(h_B(a))$ (Theorem 3.4). Rearranging the indices, if necessary, we can assume that for each $i \in \{1, 2, \ldots, n\}$,

$h_B\left(Orb_A(a_{i_A})\right) = Orb_B(a_{i_B})$ and
$h_B(a_{i_A}) = a_{i_B}$.

For each $i \in \{1, 2, \ldots, n\}$ and each $b \in Orb_B(a_{i_B})$, we have $Orb_B(b) = Orb_B(a_{i_B})$. Since X is locally connected, applying (7.2), (7.3) and the equality in part 2 of Theorem 7.1, it is true that:

$$Orb_X(a_{i_A}) = \\ \cup\left\{Orb_B(b) : B \in \mathcal{A}_c \; y \; b \in Orb_X(a_{i_A}) \cap B\right\} \\ = \cup \left\{Orb_B(b) : B \in \mathcal{A}_c \; y \; b \in Orb_B(a_{i_B})\right\} \\ = \cup_{B \in \mathcal{A}_c} Orb_B(a_{i_B}).$$

It follows that, for $i \neq j$, the orbits in X of a_{iB} and a_{jA} are different and

$$X - \{c\} = \bigcup_{B \in \mathcal{A}_c}\left(\bigcup_{i=1}^{n} Orb_X(a_{i_A})\right) \\ = \bigcup_{i=1}^{n} Orb_X(a_{i_A}).$$

Then $X - \{c\}$ is the union of exactly n orbits of X. Since $\{c\}$ is an orbit of X, X is $\frac{1}{n+1}$-homogeneous.

Now suppose that, for some $n \in \mathbb{N} - \{1\}$, X is $\frac{1}{n+1}$-homogeneous. Let $B \in \mathcal{A}_c$. Since the elements of $\mathcal{A}_c$ are homeomorphic, the only orbit of X that does not intersect B is $\{c\}$. By Lemma 7.3, the homogeneity degree of B is n. □

Corollary 7.5

Let X be a continuum with exactly one cut point c. If X is $\frac{1}{3}$-homogeneous, then each $A \in \mathcal{A}_c$ is homogeneous or each $A \in \mathcal{A}_c$ is $\frac{1}{2}$-homogeneous.

Proof

Since X is $\frac{1}{3}$-homogeneous, by Lemma 7.3, the elements of $\mathcal{A}_c$ have at most two orbits. Suppose $B \in \mathcal{A}_c$ is $\frac{1}{2}$-homogeneous. Then, there exist $b_1, b_2 \in B$ such that $Orb_B(b_1)$ and $Orb_B(b_2)$ are the two orbits of B. From part 1 of Theorem 7.1

$Orb_X(b_1) \cap B \subset Orb_B(b_1)$ and
$Orb_X(b_2) \cap B \subset Orb_B(b_2)$.

Hence, $Orb_X(b_1)$, $Orb_X(b_2)$ and $\{c\}$ are the three orbits of X. Now let $B \in \mathcal{A}_c$ and $a \in A$. Suppose without loss of generality that $a \in Orb_X(b_1)$. Then, there exists $h \in \mathcal{H}(X)$ such that $h(a) = b_1$. By Theorem 2.1, it follows that $h(A) = B$. Then A and B are homeomorphic and A is also $\frac{1}{2}$-homogeneous.

Figure 2 shows three examples of $\frac{1}{3}$-homogeneous continua with exactly one cut point. The first one is obtained by attaching, at a point, a sphere with a circumference. The second one is the union of a sequence of disks converging to a point a, and such that every two of them intersect at the cut point a. Finally, the third continuum can be seen as $(C \times D) / (C \times \{1\})$, where C is the Cantor set and D is the unit disk. Note that the last example is not locally connected, but in each case, the closure of the components of the complement of the cut point is a locally connected subcontinuum.

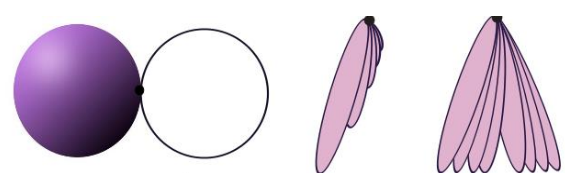

Figure 2 Examples of $\frac{1}{3}$-homogeneous continua. Own elaboration

Next, we build a well known continuum, which we will use in the following results of this section. Consider the cube $I^3 = [0,1]^3$. We divide each face of the cube into nine equal squares, this generates a subdivision of the cube into 27 equal cubes. Make a hole through the inside of each central square, this gives us a continuum M_1 formed by 20 of the 27 small cubes. Now apply this process to each one of the twenty remaining cubes; that is, we divide each face of each cube in 9 equal squares and we make a hole through the interior of the central squares, in this way we obtain a continuum $M_2 \subset M_1$. We repeat this process to obtain a nested sequence of continua $\{M_n\}_n$. Set

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$$M = \bigcap_{n\in\mathbb{N}} M_n.$$

M is a continuum called the Menger Universal Curve (Figure 3) and its name is due to the fact that it was first described by K. Menger in 1926 as a 1-dimensional continuum which contains a copy of any separable metric space of dimension 1 (Theorem 6.1, Mayer, Oversteegen and Tymchatyn, 1986, p.42). In Theorems II and III (Anderson, 1958, pp. 320-322), it is proved that M is 2-homogeneous at any of its points. From this and Theorem 4.10, it follows that M is 2-homogeneous, and by the second part of Theorem 4.3, M is homogeneous.

In the remainder of this section, the letter M will denote the Menger Universal Curve.

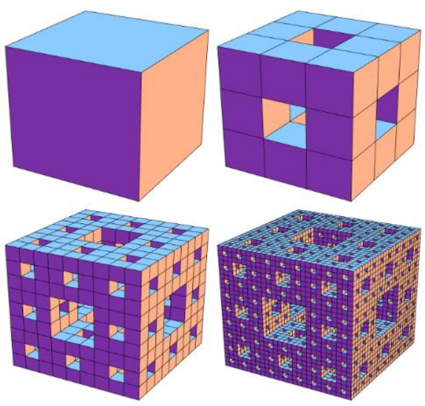

Figure 3 Construction of the Menger Universal Curve
Source:https://matemelga.wordpress.com/2015/01/09/la-esponja-de-menger/.

The proof of the following theorem can be found in Corollary 4 of (Kennedy, 1984, p.97).

Theorem 7.6
If X is any continuum, then $M \times X$ is not 2-homogeneous

Given two continuous functions $f : X_1 \to Y_1$ and $g: X_2 \to Y_2$, we define the *product function* $f \times g: X_1 \times X_2 \to Y_1 \times Y_2$, for each $(x_1, x_2) \in X_1 \times X_2$, by:

$$(f \times g)\big((x_1, x_2)\big) = (f(x_1), g(x_2))$$

Theorem 7.7
Let X be a continuum and $c \in M \times X$, then $(M \times X) - \{c\}$ is not homogeneous

Proof
Suppose that $Y = M \times X - \{c\}$ is homogeneous. Take $u, v \in Y$, then there is a homeomorphism $f : Y \to Y$ such that $f(u) = v$.

By Theorem 2.2, there is a homeomorphism $g: M \times X \to M \times X$ extending f such that $g(c) = c$. Thus, $g(u) = v$ and $g(c) = c$. Then, $M \times X$ is 2-homogeneous at c. Now, from part 1 of Theorem 4.4, $M \times X$ is homogeneous or $M \times X$ is $\frac{1}{2}$-homogeneous. If $M \times X$ was homogeneous, Theorem 4.9 would tell us that $M \times X$ is 2-homogeneous. Since this contradicts Theorem 7.6, $M \times X$ is $\frac{1}{2}$-homogeneous. From Theorem 4.4, their orbits are:

$\{c\}$ and $(M \times X) - \{c\}$.

Let $c = (a, x)$ and take $b \in M - \{a\}$. Note that (b, x) and c belong to different orbits of $M \times X$. On the other hand, M is homogeneous, so there is a homeomorphism $k: M \to M$ such that $k(a) = b$. Then $h = k \times 1_X: M \times X \to M \times X$ is a homeomorphism such that $h(c) = (b, x)$. From this contradiction we conclude that $(M \times X) - \{c\}$ is not homogeneous. □

In the following results, for a space Z, we will denote by Z^2 the space $Z \times Z$.

Theorem 7.8
Let $m > 1$ and X be a continuum m-homogeneous at the point $a \in X$. Then $Y = X^2 - \{(a, a)\}$ has at most two orbits.

Proof
Let us first show that $(X - \{a\})^2$ is contained in an orbit of Y. Take (x, y) and (u, v) in $(X - \{a\})^2$. Since X is m-homogeneous at a, by part 2 of Theorem 4.4, X is 2-homogeneous at a. Hence, there exist homeomorphisms $f, g: X \to X$ such that $f(x) = u$, $g(y) = v$ and $f(a) = a = g(a)$. Note that $f \times g: X^2 \to X^2$ is a homeomorphism such that:

$$(f \times g)\big((x, y)\big) = \big(f(x), g(y)\big) = (u, v) \text{ and}$$
$$(f \times g)((a, a)) = (a, a)$$

This shows that $(X - \{a\})^2$ is conained in an orbit of Y.

We want to see that $[(\{a\} \times X) \cup (X \times \{a\})] - \{(a,a)\}$ is contained in an orbit of Y. To this end it suffices to prove that if $x, y \in X - \{a\}$, then (a,y), (a,x), (y,a) y (x,a) belong to the same orbit of Y. Since X is 2-homogeneous at a, there is $f \in \mathcal{H}(X)$ such that $f(x) = y$ and $f(a) = a$. Set $h_1 = (1_X \times f)|_Y$, $h_2 = (f \times 1_X)|_Y$ and h_3 defined, for each $(u,v) \in Y$, by $h_{3(u,v)} = (f(v), u)$. Then, $h_1, h_2, h_3 \colon Y \to Y$ are homeomor-phisms such that:

$h_1(a,x) = (a,y)$, $h_2(x,a) = (y,a)$ and
$h_3(a,x) = (y,a)$

This proves that the four points are elements of the same orbit of Y. Since Y is the union of the sets $(X - \{a\})^2$ and $[(\{a\} \times X) \cup (X \times \{a\})] - \{(a,a)\}$, the theorem holds. □

Corollary 7.9

If $c \in M \times M$, then $(M \times M) - \{c\}$ is $\frac{1}{2}$-homogeneous

Proof

Let $a \in M$. As we mentioned before, Theorems II and III (Anderson, 1958, pp. 320-322), M is 2-homogeneous at any of its points. Now, by Theorems 7.7 and 7.8, $(M \times M) - \{(a,a)\}$ is $\frac{1}{2}$-homogeneous. Since $M \times M$ is homogeneous, there is a homeomorphism $h \colon M \times M \to M \times M$ such that $h(c) = (a,a)$. Then, $(M \times M) - \{c\}$ is homeomorphic to $(M \times M) - \{(a,a)\}$ and, therefore, is $\frac{1}{2}$-homogeneous. □

From Theorem 7.4 and Corollary 7.9 we obtain the following result.

Corollary 7.10

Let $c \in M \times M$, and X be the space obtained by pasting two copies of $M \times M$ at the point c. Then X is $\frac{1}{3}$-homogeneous, each two elements of $\mathcal{A}_c$ are homeomorphic, $\frac{1}{2}$-homogeneous and the closure of each one of them is homogeneous.

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