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Solution to the Black-Scholes equation through the Adomian decomposition method

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The Adomian Decomposition Method (ADM) is applied to obtain a fast and reliable solution to the Black-Scholes equation with boundary condition for a European option. We cast the problem of pricing a European option with boundary conditions in terms of a diffusion partial differential equation with homogeneous boundary condition in order to apply the ADM. The analytical solution of the equations is calculated in the form of an explicit series approximation with easily computable components.

Black-Schole equation, put option, call option, Adomian decomposition method.

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Introduction

In 1973 Fischer Black and Myron Scholes published a formula to find the price of financial options, which Robert Merton called the Black-Scholes equation [1,2]. For their contributions, Scholes and Merton received the Nobel Prize in Economics, Fisher Black died unfortunately he did not receive this award. [2]

The tools used to try to solve this problem are quite specialized methods and ideas in the stochastic calculus and partial differential equations. Wilmottet al. And Courtadon used finite difference methods to approximate valuation of options [3,4]. Geske and Johnson, MacMillan, Barone-Adesi and Whaley, Barone-Adesi & Adesi Elliotans Barone-developed analytical approximation methods [5,6,7,8,9]. Gülak (2010), uses the homotopyperturbation method to find an approximate equation Black-Scholes [10] solution. Cheng Zhu Liao & apply the homotopy analisys method [11] Bohner & Zheng used the Adomian decomposition method [12], however, they do not use the boundary conditions to find the approximate solution.

This article presents Adomian Decomposition Method (MDA) applied to the diffusion equation with boundary conditions of Dirichlet zero obtained by reducing the Black-Scholes equation with inhomogeneous boundary condition by changing variable is presented. The MDA is intended to provide an analytical solution to an equation or a system of differential equations.

The method is based on considering the decomposition of the unknown function in an infinite series $\sum_{n=0}^{\infty} u_n$, and decomposing the nonlinear term of the equation in a set, $\sum_{n=0}^{\infty} A_n$, where A_n are called Adomian polynomials. It had its beginnings in the 80's, when George introduced and developed the Adomian decomposition method called, for solving linear and nonlinear equations for both ordinary differential equations to partial differential equations [13]. The method has been applied in several deterministic and stochastic linear and nonlinear problems in physics, biology, chemistry and economics [12,14,15,16].

Buying and selling options and Black-Scholes equation

Consider the problem of finding the price of an option (a certain coin) to mature at time T, with cost K. The option price can be thought of as paying a premium for the right to exercise the option to expiration time. The problem is to find the "right" price of the option. To find a solution to the problem should be considered the primary financial markets, for example, the randomness features, it is not known how the currency will be worth over time [2]. In general, we have the following cases [17,18]: Suppose that an option at time t = 0, which gives the right to buy a share of stock to or at time T, is acquired maturation time or time expiration of the option.

If the option is exercised at a fixed price K, called the exercise price of the option, only the maturation time T, then the option is known as a European call option.

If the option can be exercised until or at time T, is called American call.

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The holder of an option to purchase is not required to perform this, so if the time *T*, the price X_t is less than *K*, the option holder may buy one share for X_t in the market, and so the ticket will expire as worthless treatment. If the price X_t exceeds *K*, good choice would exercise the call, ie, you can buy the stock at the price K, and sell precioXt for a net profit of $X_t - K$. Therefore, the buyer of the European call option is entitled to a fee of

$$f_0 = máx(0, X_T - K) = \begin{cases} X_T - K, si & X_T > K, \\ 0 \quad , si & X_T \leq K. \end{cases}$$

The put option is an option to sell a stock at the price K or given to a particular maturity date T. A European put option is exercised only at the time of maturation, American put option can be exercised until or while T. The buyer of a European put option makes a profit

$$f_0 = máx(0, K - X_T) = \begin{cases} K - X_T, si & X_T < K, \\ 0 \quad , si & X_T \geq K. \end{cases}$$

In financial mathematics, it can be shown that to study a self-financing strategy can be reached at the following partial differential equation called the Black-Scholes equation [1.17].

$$rf(t,x) = f_t(t,x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) + rxf_x(t,x), \quad x > 0, t \in [0,T], \qquad (1)$$

Where, x represents the value of the asset, *t* the time, *f* the option price, r is the interest rate of the debt market and σ is the volatility of the stock, measured as the standard deviation of the logarithms of the share price. Henceforth, we will work with European options.

The boundary conditions necessary to solve the equation (1) for options for European sales are [18]:

- **Frontier in *x=0***. Consider what happens when x = 0, if x = 0 to time t_0, the geometric Brownian motion implies that x = 0 to any $t \geq t_0$, according to f_0, payment of option exercise time T is K, in addition, considering that there is no arbitrage, then the change of the K considering the discount that would have the cash rate risk free rate over time t can be approximated by $Ke^{-r(T-t)}$. Therefore, f (0,t) = $Ke^{-r(T-t)}$.

- **Frontier in *x=T***. Consider, if x takes very large values, it is almost certain that the exercise of the option is not paid, then we say that $\lim_{x\to\infty} f(x,t) = 0$, therefore, it is consider f(*T*,t) ≈ 0.

Similarly, in the case of the purchase option, the analysis for the boundary conditions is:

- **Frontier in *x=0***. If x=0, exercise payment at maturity is clearly zero, and thus, $f(0,t) = 0$.

- **Frontier in *x=T***. In the limit, when $x \to \infty$, the case is more complicated. If *x* takes a big value, the purchase option is exercised and a gain is X_T - *K at time* T. The option value may be approximated as x - $Ke^{-r(T-t)}$, when $x \to \infty$. However, it can be neglected $Ke^{-r(T-t)}$ because x takes very large values, and thus, it can be said that f(x,t) ~ x for all t, i.e, $lim_{x\to\infty} \frac{f(x,t)}{x} = 1$. Therefore, f(*T*,t) ≈ x - $Ke^{-r(T-t)}$, or, f(*T*,t) ≈ x.

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Another condition used for borders of any kind of option has been f (x,t)~$e^{-r(T-t)}f_0(xe^{-r(T-t)})$, which has had economic interpretation in the sense that if the stock price were deterministic, the price of the shares of x at time t could be approximated by $xe^{-r(T-t)}$, and the resulting option value considering the appropriate discount rate for the risk free rate. This formula provides the conditions for European sales and purchases, with the restriction to f(x,t) ≈ x - $Ke^{-r(T-t)}$ when $x\to\infty$, for European purchases [18].

There are other expressions to approximate the limiting behavior of European options, Schwartz uses $\lim_{x\to\infty}\frac{\partial f(x,t)}{\partial x}=1$, however, Persson and von Sydow, assume that the option price is linear with respect to x at the borders to any option, then $\frac{\partial^2 f(x,t)}{\partial^2 x}=0$, in the frontiers [18].

The Adomian Decomposition Method

The Adomian decomposition method to find an analytical solution in series form [12-16] and consists of identifying the equation given in linear and non-linear parts, and invest the higher order differential operator that is in the linear part , consider the unknown function as a series whose components are well defined, then the nonlinear function is decomposed in terms of Adomian polynomials. We define the initial conditions and / or boundary and the terms involving the independent variable as initial approximation. Thus is successively series terms solution by a recurrence relation.

In general, the scheme is as follows: given a differential equation,

$$Fu(t)=g(t) \tag{2}$$

Where F represents a non-linear differential operator that encompasses both linear and nonlinear terms. Then equation (2) can be written as,

$$Lu(t)+Ru(t)+Nu(t)=g(t) \tag{3}$$

Where in the linear operator is L + R, L is an easily invertible operator and the remaining R linear operator, N denotes the nonlinear operator and manage the independent function u (t).

Resolving to Lu(t),

$$Lu(t)=g(t)-Ru(t)-Nu(t)$$

L is invertible, operating with reverse L^{-1} we have that,

$$L^{-1}Lu(t)=L^{-1}g(t)-L^{-1}Ru(t)-L^{-1}Nu(t)$$

An equivalent expression is

$$u(t)=\varphi+L^{-1}g(t)-L^{-1}Ru(t)-L^{-1}Nu(t) \tag{4}$$

Where φ is the constant of integration and satisfies L_φ = 0. For problems with initial value in t = a, have conveniently defined L^{-1} to $L=\frac{d^n y}{dx^n}$ as *nth* defined integration aat.

This method assumes a solution in the form of infinite series for the unknown function u (t) given by,

$$u(t)=\sum_{i=0}^{\infty}u_i(t) \tag{5}$$

The nonlinear term Nu (t) is decomposed as

$$Nu(t)=\sum_{n=0}^{\infty}A_n(u_0,u_1,\dots,u_n) \tag{6}$$

Where A_n is called Adomian polynomial, and depends on the characteristic of the nonlinear operator. The A_n 's are calculated in a general way by the formula:

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{j=0}^{\infty} \lambda^j u_j\right)\Big|_{\lambda=0} \quad (7)$$

The formula (7) is easy to code in a software as MATLAB or MAPLE [19].

Substituting the expressions given by (5), (6) and (7) into equation (4) have,

$$\sum_{i=0}^{\infty} u_i(t) = \varphi + L^{-1}g(t) - L^{-1}R\sum_{i=0}^{\infty} u_i(t) - L^{-1}\sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n)$$

Consequently we obtain

$$\begin{cases} u_0(t) = \varphi + L^{-1}g \\ u_{n+1}(t) = -L^{-1}Ru_n(t) - L^{-1}A_n(u_0, u_1, \dots, u_n) \end{cases} \quad (8)$$

The solution in practice is given by the k-th approximation ψ_k:

$$\psi_k = \sum_{i=0}^{k-1} u_i(t) \quad (9)$$

The decomposition of the solution in series, usually converges fast. The speed of convergence causes the need of few terms. The conditions for which the MDA converges has been strongly studied by Cherruault [20], Adomian Cherruault [21], and abbaoui and Cherruault [22,23].

Solutions of the Black-Scholes equation via MDA

Direct application of MDA

Consider equation (1) with terminal function $f(T,x) = f_T$, which, f_T =max (x – K,0), if it is a purchase option, o f_T =max (K – x,0), if it is an option. You can apply the MDA considering operators as mentioned by [12]

$$L = (.)_t, \quad R = \frac{1}{2}\sigma^2 x^2 (.)_{xx} + rx(.)_x - r(.), \quad N = 0, \quad y \quad g = 0.$$

Rewriting the equation (1)

$$f_t(t,x) = -\frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) - rxf_x(t,x) + rf(t,x),$$

Applying $L^{-1} = \int_{s=t}^{s=T} (.)\, ds$ we obtain,

$$L^{-1}f_t(t,x) = -\frac{1}{2}\sigma^2 L^{-1}x^2 f_{xx}(t,x) - rL^{-1}xf_x(t,x) + rL^{-1}f(t,x),$$

$$f(T,x) - f(t,x) = -\frac{1}{2}\sigma^2 \int_{s=t}^{s=T} x^2 f_{xx}(s,x)\,ds - r\int_{s=t}^{s=T} xf_x(s,x)\,ds + r\int_{s=t}^{s=T} f(s,x)\,ds$$

$$f(t,x) = f_T + \frac{1}{2}\sigma^2 \int_{s=t}^{s=T} x^2 f_{xx}(s,x)\,ds + r\int_{s=t}^{s=T} xf_x(s,x)\,ds - r\int_{s=t}^{s=T} f(s,x)\,ds$$

Assuming that the solution can be expressed in terms of an infinite series,

$$f(t,x) = \sum_{i=0}^{\infty} f_i(t,x)$$

We obtain,

$$\sum_{i=0}^{\infty} f_i(t,x) = f_T + \frac{1}{2}\sigma^2 \int_{s=t}^{s=T} x^2 \sum_{i=0}^{\infty} f_{ixx}(t,x)\,ds + r\int_{s=t}^{s=T} x\sum_{i=0}^{\infty} f_{ix}(t,x)\,ds - r\int_{s=t}^{s=T} \sum_{i=0}^{\infty} f_i(t,x)\,ds$$

$$\sum_{i=0}^{k} f_i(t,x) = f_T + \frac{1}{2}\sigma^2 \sum_{i=0}^{k} \int_{s=t}^{s=T} x^2 f_{ixx}(t,x)\,ds + r\sum_{i=0}^{k} \int_{s=t}^{s=T} xf_{ix}(t,x)\,ds - r\sum_{i=0}^{k} \int_{s=t}^{s=T} f_i(t,x)\,ds$$

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And each term of the approximation is represented determined by

$$\begin{cases} f_0(t,x) = f_T \\ f_{n+1}(t,x) = \frac{1}{2}\sigma^2 \int_{s=t}^{s=T} x^2 f_{nxx}(t,x)\,ds + r\int_{s=t}^{s=T} x f_{nx}(t,x)\,ds - r\int_{s=t}^{s=T} f_n(t,x)\,ds. \end{cases} \quad (10)$$

To n>0. Note that for the (k +1)-th approximation ψ_{k+1} truncating the serie solution f (t,x) with k+1 terms is given by:

$$f(t,x) \approx \psi_{k+1} = \sum_{i=0}^{k} f_i(t,x) \quad (11)$$

Note that in no point boundary conditions were used, therefore, when considering the problems of boundary conditions will be given special treatment, as were discussed below.

Reduction of the problem of option to broadcast problem

Now, consider the initial value problem and boundary conditions for the valuation of the option (CallOption), C (t, x),

$$\begin{cases} rC(t,x) = C_t(t,x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t,x) + rxC_x(t,x), & x > 0, t \in [0,T] \\ C(T,x) = máx(x-K,0), & \\ C(t,x) = x - Ke^{-r(T-t)}, & \text{cuando } x \to \infty \\ C(t,0) = 0, & \forall t > 0. \end{cases} \quad (12)$$

To reduce the problem (12) to a diffusion problem the following changes of variable as presented in [17,24] are used,

$$\tau = \frac{1}{2}\sigma^2(T-t), \quad (13)$$

$$y = ln\left(\frac{x}{K}\right), \quad (14)$$

$$C(t,x) = KV(\tau,y), \quad (15)$$

Reducing equation (12)

$$-V_\tau(\tau,y) + V_{yy}(\tau,y) + \left(\frac{2r}{\sigma^2} - 1\right)V_y(\tau,y) = \frac{2r}{\sigma^2}V(\tau,y) \quad (16)$$

With initial condition

$$V(0,y) = \frac{1}{K}C(T,x) = \frac{1}{K}máx(x-K,0) = \frac{1}{K}máx(Ke^y - K,0) = máx(e^y - 1,0)$$

Now, using the following transformation:

$$U(\tau,y) = e^{ay+b\tau}V(\tau,y)$$

Taking $a = \frac{1}{2}\left(\frac{2r}{\sigma^2} - 1\right)$ y $b = (1+a)^2$, then the problem (12) becomes

$$\begin{cases} U_t(\tau,y) = U_{yy}(\tau,y), \quad y > 0, t \in [0,T] \\ U(0,y) = máx\left(e^{\frac{1}{2}(\gamma+1)y} - e^{\frac{1}{2}(\gamma-1)y}, 0\right), \\ U(\tau,L) = e^{\frac{1}{2}(\gamma+1)L + \frac{1}{4}(\gamma+1)^2\tau} - e^{\frac{1}{2}(\gamma-1)L + \frac{1}{4}(\gamma-1)^2\tau}, \\ U(\tau,0) = 0, \quad \forall\tau > 0. \end{cases} \quad (17)$$

Thus, the solution to the partial differential equation Black-Scholes transforming a diffusion equation is given by

$$C(t,x) = Ke^{-\frac{1}{2}(\gamma+1)y - \frac{1}{4}(\gamma-1)^2\tau}U(\tau,y), \quad (18)$$

Where τ, y are given by (13) and (14).

Reducing the problem to puttable diffusion problem

Now, consider the problem of Black-Scholes initial value and boundary conditions for the valuation of the put option, Put Option, P (t, x), given by

$$\begin{cases} rP(t,x) = P_t(t,x) + \frac{1}{2}\sigma^2 x^2 P_{xx}(t,x) + rxP_x(t,x), & x > 0, t \in [0,T] \\ P(T,x) = máx(K-x,0), & \\ P(t,x) = Ke^{-r(T-t)} - x, & \text{cuando } x \to 0 \\ P(t,x) = 0, & \text{cuando } x \to \infty, t \in [0,T]. \end{cases} \quad (19)$$

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It can reduce the problem (19) to a diffusion problem using variable changes (13), (14) and (15), similar to equation (16) is obtained,

$$-V_\tau(\tau,y)+V_{yy}(\tau,y)+\left(\frac{2r}{\sigma^2}-1\right)V_y(\tau,y)=\frac{2r}{\sigma^2}V(\tau,y) \qquad (20)$$

With initial condition

$$V(0,y)=\frac{1}{K}P(T,x)=\frac{1}{K}\text{máx}(K-x,0)=\frac{1}{K}\text{máx}(K-Ke^y,0)=\text{máx}(1-e^y,0)$$

Then a transformation is applied as follows,

$$U(\tau,y)=e^{ay+b\tau}V(\tau,y)$$

Analogous to the previous case, if $a=\frac{1}{2}\left(\frac{2r}{\sigma^2}-1\right)$ y $b=(1+a)^2$, then the problem (19) becomes

$$\begin{cases} U_\tau(\tau,y)=U_{yy}(\tau,y), & y>0, t\in[0,T] \\ U(0,y)=\text{máx}\left(e^{\frac{1}{2}(\gamma-1)y}-e^{\frac{1}{2}(\gamma+1)y},0\right), \\ U(\tau,L)=e^{\frac{1}{2}(\gamma-1)L+\frac{1}{4}(\gamma-1)^2\tau}-e^{\frac{1}{2}(\gamma+1)L+\frac{1}{4}(\gamma+1)^2\tau}, \\ U(\tau,0)=0, & \forall\tau>0. \end{cases} \qquad (21)$$

Therefore, the solution to the partial differential equation of Black-Scholes transforming a diffusion equation is given by

$$P(t,x)=Ke^{-\frac{1}{2}(\gamma-1)y-\frac{1}{4}(\gamma+1)^2\tau}U(\tau,y), \qquad (22)$$

Where τ, y are given by (13) and (14).

Applying MDA to European options

Given the system

$$\begin{cases} u_\tau(\tau,y)=u_{yy}(\tau,y), & y>0, t\in[0,T] \\ u(0,y)=u_0(y). \end{cases} \qquad (23)$$

Following the MDA algorithm, considering

$$L=\frac{du}{d\tau},\ R=\frac{d^2u}{dx^2},\ N=0,\ \text{y}\ g=0,$$

Then we obtain

$$L^{-1}u_\tau(\tau,y)=L^{-1}u_{yy}(\tau,y),$$
$$u(\tau,y)=u(0,y)+\int_{s=0}^{\tau}u_{yy}(s,y)ds$$

Assuming a solution as infinite series $u(\tau,y)=\sum_{i=0}^{\infty}u_i(\tau,y)$, we obtain,

$$\sum_{i=0}^{\infty}u_i(\tau,y)=u(0,y)+\int_{s=0}^{\tau}\sum_{i=0}^{\infty}u_{iyy}(s,y),ds$$

For an approach with k +1 terms we have,

$$\sum_{i=0}^{k}u_i(\tau,y),=u(0,y)+\int_{s=0}^{\tau}\sum_{i=0}^{k}u_{iyy}(s,y),ds$$
$$\sum_{i=0}^{k}u_i(\tau,y),=u(0,y)+\sum_{i=0}^{k}\int_{s=0}^{\tau}u_{iyy}(s,y)ds$$

Therefore, the (k +1)-th approximation to the solution is given by,

$$\psi_k(\tau,y)=\sum_{i=0}^{k-1}u_i(\tau,y)\approx u(\tau,y) \qquad (24)$$

Thus, the solution to the original problem is given for the purchase option for

$$C(t,x)=Ke^{-\frac{1}{2}(\gamma+1)y-\frac{1}{4}(\gamma-1)^2\tau}\psi_k(\tau,y), \qquad (25)$$

And for the put option,

$$P(t,x)=Ke^{-\frac{1}{2}(\gamma-1)y-\frac{1}{4}(\gamma+1)^2\tau}\psi_k(\tau,y), \qquad (26)$$

Where τ, y are given by (13)and (14).

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Solution of the diffusion equation with boundary conditions by MDA

The Adomian decomposition method is not suitable for solving partial differential equations with inhomogeneous boundary conditions but under a change of variables can transform the initial value problem and inhomogeneous boundary conditions to an initial value problem with conditions border homogeneous and mention Adomian & Rach in 1992 and Lou et al.en 2006 [25,26].

Transforming the original problem by following the methodology presented by Lou et al., Assume that

$$U(\tau,y)=u(\tau,y)+w(\tau,y) \qquad (27)$$

con

$$w(\tau,y)=U(0,y)+\left(U(0,y)-U(\tau,L)\right)\left(\frac{y-y0}{L-y0}\right).$$

Therefore, the problems (17) and (21) acquire a general way,

$$\begin{cases} u_\tau(\tau,y)=u_{yy}(\tau,y)-w_\tau(\tau,y), & y>0, t\in[0,T] \\ u(0,y)=u_0(y)-w(0,y), & \\ u(\tau,L)=0, & \\ u(\tau,0)=0, & \forall\tau>0. \end{cases} \qquad (28)$$

Where $u_0(y)=\text{máx}\left(e^{\frac{1}{2}(\gamma+1)y}-e^{\frac{1}{2}(\gamma-1)y},0\right)$ if it is an option, or $u_0(y)=\text{máx}\left(e^{\frac{1}{2}(\gamma-1)y}-e^{\frac{1}{2}(\gamma+1)y},0\right)$, if the problem corresponds to a put option.

Now, following the MDA algorithm, consider, $L=\frac{du}{d\tau} R=\frac{d^2u}{dx^2}, N=0,$ and $g=-w_t(\tau,y).$. Thus

$$L^{-1}u_\tau(\tau,y)=-L^{-1}w_\tau(\tau,y)+L^{-1}u_{yy}(\tau,y),$$
$$u(\tau,y)=u(0,y)-L^{-1}w_\tau(\tau,y)+\int_{s=0}^{\tau}u_{yy}(s,y)ds.$$

Considering a solution as infinite series $u(\tau,y)=\sum_{i=0}^{\infty}u_i(\tau,y),$ we obtain,

$$\sum_{i=0}^{\infty}u_i(\tau,y)=u(0,y)-L^{-1}w_\tau(\tau,y)+\int_{s=0}^{\tau}\sum_{i=0}^{\infty}u_{iyy}(s,y)\,ds.$$

For an approach with k +1 terms,

$$\sum_{i=0}^{k}u_i(\tau,y)=u(0,y)-L^{-1}w_\tau(\tau,y)+\int_{s=0}^{\tau}\sum_{i=0}^{k}u_{iyy}(s,y)\,ds,$$
$$\sum_{i=0}^{k}u_i(\tau,y)=u(0,y)-L^{-1}w_\tau(\tau,y)+\sum_{i=0}^{k}\int_{s=0}^{\tau}u_{iyy}(s,y)ds.$$

And the terms of the series are completely determined by

$$\begin{cases} u_0(\tau,y)=u(0,y)-L^{-1}w_\tau(\tau,y), \\ u_{n+1}(\tau,y)=\int_{s=0}^{\tau}u_{iyy}(s,y)ds. \end{cases} \qquad (29)$$

Therefore, the (k +1)-th approximation to the solution is given by,

$$u(\tau,y)\approx\psi_k(\tau,y)=\sum_{i=0}^{k-1}u_i(\tau,y) \qquad (30)$$

The solution to the original problem is given for the purchase option for

$$C(t,x)=Ke^{-\frac{1}{2}(\gamma+1)x-\frac{1}{4}(\gamma-1)^2\tau}(\psi_k+w)(\tau,y), \qquad (31)$$

And the option of selling linear regression

$$P(t,x)=Ke^{-\frac{1}{2}(\gamma-1)x-\frac{1}{4}(\gamma+1)^2\tau}(\psi_k+w)(\tau,y). \qquad (32)$$

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Discussion, results and simulations

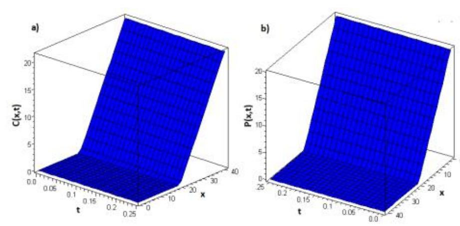
Under the conditions of a fluctuating market, the best choice to simulate the behavior of the costs of stock options or European sales, we consider the problem of the Black-Scholes equation with inhomogeneous boundary conditions because they resemble reality market, since it involved the maturation time and considerations on the rate of risk free rate. Simulations presented in this section to value options for buying and selling using the problems (12) and (19) with values of $r = 0.05, \sigma = 0.317, K = 20,\ T = 0.25$ *(3 months).*

MDA was directly applied to equation (1), ie, the algorithm was applied to the Black-Scholes equation using only the initial condition f_0, this methodology has been presented in the literature by Bohner and Zheng (2009) [12]. The results of the simulations are presented in Graphic 1. a) the behavior that would have the cost of the call option over time and can be seen in Graphic 1 a) the solution of the put option is presented.

When considering the boundary conditions of the problem (1), ie, working on problems (12) and (19).

For the option of buying and selling, respectively, can follow the methodology presented in this article, which is to create a new problem of differential equations reduced to a diffusion equation, and applying the MDA directly using only initial condition. Results vary with respect to the solution of the above problem (Graphic 1), the approximation to the solutions of problems (12) and (19) are presented in Graphic 2.

Approximate solution by applying the MDA directly with k = 10 to the Black-Scholes equation without boundary conditions parameterized $r = 0.05, \sigma = 0.317, K = 20$, $T = 0.25$ (3 months) to the purchase option a) and sell option b).



Graphic 1

Now, to use the boundary conditions of the diffusion equation in (17) and (21) for problems related to (12) and (19), we considered the decomposition of the unknown solution into two parts, one was unknown and therefore would be approximated by the MDA, and another that is clearly determined by the boundary conditions,$w(\tau, y)$. With this methodology, a new diffusion problem with homogeneous boundary conditions is obtained.

The results of the simulations of the system (28) given by the expressions (31) and (32) is located in Graphic 3 and 4, therein, the absolute errors are also appreciated from the simulations obtained by applying the equation considering MDA diffusion without using boundary conditions with respect to the simulations obtained using the boundary conditions (Graphic 3d) and 4d)).

Approximate solution by applying the MDA to the diffusion equation (23) with k = 10 I obtained from the Black-Scholes equation without boundary conditions parameterized $r = 0.05, \sigma = 0.317, K = 20,\ T = 0.25$ (3 months) for the purchase option) and put option c), the solutions according to formulas (25) and (26) are shown in b) and d), respectively.

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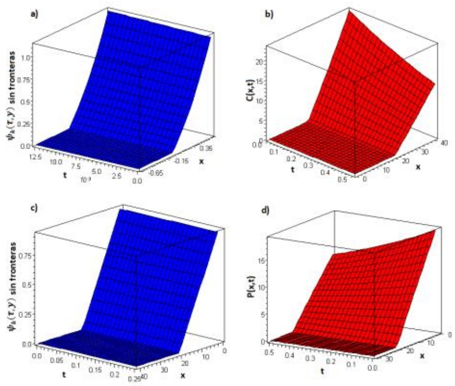
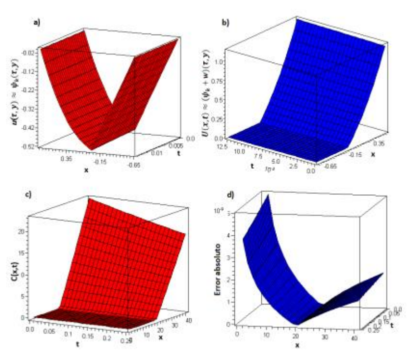


´ **Graphic 2**

Errors in the approximations could lead to potential cash losses when working with high prices in the options.

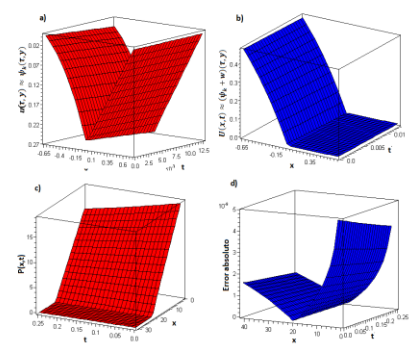
Therefore, the best choice is to consider the whole set of conditions or assumptions necessary for the Black-Scholes model reflects the reality in the prices for items in the financial market. Therefore, the MDA is an alternative to solve the problem of Black-Scholes equation by making appropriate changes variable.

a) Applying the MDA approximate solution to the diffusion equation with Dirichlet boundary conditions of zero according to equation (28) with k = 10, obtained from the Black-Scholes problem with parameters r = 0.05, σ = 0.317, K = 20 , T = 0.25 (3 months) of the problem (17). In b) the solution to (17) using equation (27) is presented. The approximate solution according to formula (31) is shown in c) and d) the graph of the error between approximations obtained by the MDA with and without using the boundary conditions of the problem (17 presented), see Graphic 2.



Graphic 3

a) Applying the MDA approximate solution to the diffusion equation with boundary conditions of Dirichlet zero according to equation (28) with k = 10, obtained from the Black-Scholes problem with parameters $r = 0.05, \sigma = 0.317, K = 20,\ T = 0.25$; (3 months) of the problem (21). In b) the solution to (21) using equation (27) is presented. The approximate solution according to formula (32) is shown in c) and d) the graph of the error between approximations obtained by the MDA with and without using the boundary conditions of the problem (21 presented), see Graphic 2.



Graphic 4

Conclusions

The Adomian decomposition method for its rapid convergence Cherruault as reported [20], and Adomian Cherruault [21], and Cherruault abbaoui [22,23] make it an alternative and effective tool to solve the problem of the equation of Black-Scholes.

The results obtained to reduce the original equation to a diffusion equation with zero Dirichlet conditions for using the boundary conditions show the effectiveness of the method. Thus, the methodology presented in this article can be very useful to the people paying the price of the options.

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