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Mandelbrot sets in diversifying markets with Julia sets

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In this article an analysis of one of the so many slopes of the geometry fractal was made applied to the financial market, that given its dynamic and volatile system, is resembled the variations of a fractal.
With base in the sets of Mandelbrot and Julia, a model from application to the financial market of capitals at local and international level with the purpose was developed of obtaining short term prognoses of the prices of the actions for the decision making.

Fractal, iteration, numbers complex, disturbance

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Introduction

One of the mathemathical advances with higher complexity, for leaving standards that the mathematician community had implemented for many centuries, is the fractal geometry, developed by the polish mathematician Benoit Mandelbrot (1924-) in 1975. It was motivated to replicate irregular or "natural" forms that Euclidean geometry could not represent, with the purpose of understanding disordered structures and its formation by random processes.

Mandelbrot did so through what he called "fractals", name inspired in the Latin adjective "fractus" which means fragmented. Benoit Mandelbrot postulated fractal geometry in various literary works[41], based on the ideas of two French mathematicians: Gaston Maurice Julia (1893-1978) and Pierre Joseph Fatou (1878-1929), who following an iterative process over the numbers field created the well-known Julia sets and its complement, the Faotu sets; at the same time, Mandelbrot worked specially over a subset of the Julia set, that have his name.

Fractals have two properties that highlight their essence: self-similarity and fractal dimension, acquired by its scaled invariance, this is, the more its scale expands, they do not lose their original shape. The fractal concept is used to identify order in many problems of non-linear characteristics, in fact, without the help of fractals, complex systems cannot be designed in detail.

[41] Benoit Mandelbrot, "*The fractal geometry of nature",* Freeman, New York, 1982; y, Benoit Mandelbrot , "*Fractals; Form, Chance and Dimension"*, Freeman, New York, 1977

Many applications of fractal geometry have emerged thanks to its ability to identify, quantify and analyze repetitive patterns; the objective of this work is to develop its application in the Financial Market, since this is a dynamic system, because it contains variables with the same characteristics of fractals. Fractal geometry is used to enlarge small variations or fluctuations of a time series using iterative processes, thus creating the qualitative large-scale changes.

Mandelbrot himself directed fractal geometry to this field[42] by analizing cotton price variations, because this variable have a non-linear dynamic; he found that movement curves of the prices in different times have the same form, facilitating the prediction.

Afterwards, Mandelbrot delved into financial subjects relative to temporal variability of speculative prices in his work "Fractal and Scaling in Finance" in 1997; because financial markets develop between chaos and order, where initial small changes produce large changes in the movements of consumer prices. This property is the main idea for the development of mathematical models that provide short-term forecasts by modeling the behavior of prices, in order to make the best possible decisions (purchase or sale)

Fractal analysis is linked to the Chaos theory because it recognizes that not all studied models are linear. Such is the case of the models used to analyze financial markets; since they are nonlinear dynamic systems that may change the initial values by interacting with past or external values, causing totally different results to those expected.

[42] Benoit Mandelbrot,*" Fractal and Scaling in Finance"*, Springer, New York, 1997, 551 p

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Iterative process of Nonlinear Functions

In simple terms, a fractal is a geometric form repeating itself at any scale in which it is observed. Rigorously, a fractal is the final result of the infinite iteration of a determined geometric process, in particular, resulting from the composition of functions of a quadratic function on a complex field. First, we will define the characteristic properties of fractals, then delving into its construction.

Fractal properties:

Self-similarity: it means that each fragment of the object have the same characteristics of the complete figure, and can be repetead infinitely; they are based on complex numbers.

There are two types of fractals:

Linear: they are the same at different scales and therefore tend to infinity.

Non-linear: emerge from complex distortions, and are found in nature.

Fractal dimension: reveals that the dimension of the fractal do not correspond to an integer but to a fraction.

Because the theory of fractals comprises the field of complex numbers, we give a brief introduction to them.

Comment 1

The set formed by the numbers of the form $a+bi$ with $a,b\in\Re$, $i=\sqrt{-1}$ are known as complex numbers.

If $z=a+bi$ is a complex number. We will call the real part of z to the real a, and the imaginary part of z to the real b

The complex number field is denoted by C. We mention the most important properties of complex numbers:

- The modulus of z, denoted by $|z|$ equals $|z|=\sqrt{a^2+b^2}$

- Where $z=a+bi$ and $x=c+di$

Then:

Sum:

$$z+x=(a+c)+i(b+d)$$

Product:

$$zx=(ab-cd)+i(ac+bd)$$

- The conjugate of a complex number $z=a+bi$, is denoted as $\bar{z}$ and corresponds to $\bar{z}=a-bi$

- Exponential notation of z

$$z=|z|\exp(i\theta) \text{ with } 0\le\theta\le 2\pi$$

- Polar notation of z

$$z=|z|(\cos(\theta)+i\,\mathrm{sen}(\theta))$$

with $0\le\theta\le 2\pi$

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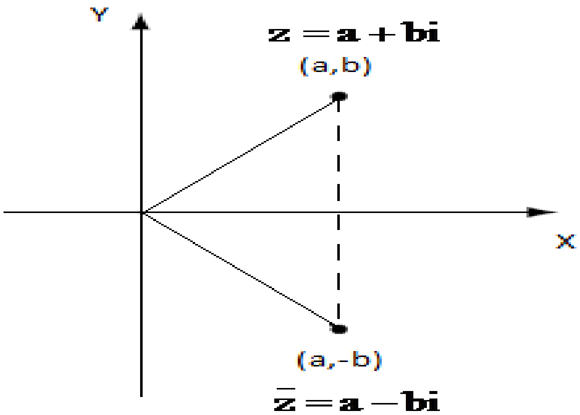
Complex plane



Graphic 1

From the geometric point of view, complex numbers can be identified with the points of the carthesian plane making the point (a,b) correspond to the complex $z = a + bi$, as shown in the graphic 1.

The topology of the complex plane can be conceived through the equivalence established between the Riemman sphere and the complex plane, this is, the projection of the points of the sphere of unitary radius centered at N, tangent to the complex plane, over it, following a bijection application. In Graphic 2 we can visualize this projection.

The origin of the fractal geometry theory dates back to the work of French mathematicians Gaston Julia (1893-1978) and Pierre Fatou (1878-1929) who in their manuscripts postulated the sets that bear their names, within a complex dynamic system, observing the behavior of an orbit of a point c over the extended complex plane, defined as $\hat{C} = C \cup \{\infty\}$, applying iterationally a quadratic function f.

Despite being the precursors of the fractal theory, Fatou and Julia could not see graphically their sets due to the difficulty of the arithmetic calculus.

Comment 2

A discrete dynamical system is a pair $(\mathbf{X}, f)$ where $\mathbf{X}$ is a field and $f : \mathbf{X} \to \mathbf{X}$. Given a point $x \in \mathbf{X}$, the set $\{x, f^1(x), f^2(x), f^3(x), f^4(x), \ldots\}$ will be called the orbit of x, where $f^n(x) = f \circ \cdots \circ f(x)$.

The point $x \in \mathbf{X}$ that satisfies $f(x) = x$ is called fixed point or "point of balance" of the function f.

The point $x \in \mathbf{X}$ that satisfies $f^n(x) = x$ and $f^i(x) \neq x$ with $i > n$ is called periodic point of the function f of period $n > 1$.

Making $\mathbf{X} = C$, the classification of fixed points according to their properties in a complex dynamic system (C, f), are the following:

- $z_0 \in C$, z_0 is an attractor point if $$|f'(z_0)| < 1 \quad (1.1.1)$$

- $z_0 \in C$, z_0 is a repeller point if $$|f'(z_0)| > 1 \quad (1.1.2)$$

- $z_0 \in C$, z_0 is an indifferent point if $$|f'(z_0)| = 1 \quad (1.1.3)$$

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- $z_0 \in C$, z_0 is a super attractor point if $\left|f^{'}(z_0)\right| = 0$ $\qquad (1.1.2)$

Comment 3

It is said that a dynamic system is stable if the dynamic does not change under small perturbations of f .

After classifying fixed points on (C, f), let us define the Julia sets.

Julia sets and FATOU set

Julia sets occur in complex dynamic systems, like an example we have physical, biological, computational, social and economics systems. Such systems generally exhibit invariance at different scales, because their behavior does not change by the rescaling of the space-time variables that define and govern their dynamic.

Julia and Fatou by 1918, implemented the iterative process under the transformation $f_c(z) = z^2 + c$, to obtain the orbit of the z point.

Comment 4

If (C, f) is a complex dynamic system, with $f_c : C \to C$ such that $f_c(z) = z^2 + c$, with $c, z \in C$, the Julia set under f_c is given by:

$$J_c(f) = \text{clausura}\{z \in C \text{ tq } z \text{ es un punto periódico repulsor de } f\} \quad (1.2.1)$$

Comment 5

The closure of Ω is defined as the intersection of all closed sets which contain Ω. The Julia set by definition, are all points that divide $\hat{C}$ in two parts, those that, following the iterative process, have an inestable orbit and those for which its orbit converges.

Iterative Process on C

Iterative process for computing the orbit of a point $z_0 \in C$ in (C, f)

if $f : C \to C$ so then $f = z^2 + c$ with $c \in C$,

for $n = 1, 2, \ldots$, then :

$$f^1(z_0) = z_0^2 + c$$

$$f^2(z_0) = f \circ f(z_0)$$

$$\vdots$$

$$f^n(z_0) = f \circ \cdots \circ f(z_0)$$

Graphic 2

For the particular case $f_c(z) = z^2$, the periodic points of the period $n > 1$ are those such that $f_c^n(z) = z^{2^n} = z$, hence $z^{2^n - 1} = 1$, equivalent to $|z| = 1$, then:

$$\left|\left(f^{n}(z)\right)\right| = \left|2^n\left(z^{2^n-1}\right)\right| = 2^n\left|z^{2^n-1}\right| = 2^n(1) > 1$$

It therefore appears that all periodic point of $f_c(z) = z^2$ of period $n > 1$ is repellent, then the Julia set is given by:

$$J_c(f) = \text{cerradura}\{z \in C \text{ t.q. } |z| = 1\}$$

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From this reasoning, we can find the repellent periodic p-points, that belong to the Julia set, this is, we have to find the roots of polynomials of degree 2^p, as for $f_c^2(z)$ with f_c a quadratic function, is a polynomial of degree 4, for is a polynomial of degree 8, and so on; then, find the condition that must be fulfilled by z, in order to be a repellor point; this proceeding difficults the construction of a Julia set, because of the arithmetic calculus it has.

The following theorema provides an alternative way of calculating the Julia set, in a simple way:

Theorem 1

Sí $z \in J_c(f)$, then:

$$J_c(f) = \text{cerradura}\left(\bigcup_{n=0}^{\infty} f_c^{-n}(z)\right) \qquad (1.2.2)$$

This theorema states that $J_c(f)$ is an attractive set of f^{-1}, and provides a simple algorithm to plot the Julia set through computational calculations, in the following way:

The inverse of the function f_c^k for $k \to \infty$ must be calculated, valued at a fixed repellor point, i e , for a $z_0 \in C$ resulting $f_c(z_0) = z_0$ and meet $\left| f_c^{'}(z_0) \right| > 1$; by definition $z_0 \in J_c(f)$, then is $Z_0 = \{ z_0 \}$

We iterate the function $f_c(z_0)$, with the aim that in the k-th step, the set $Z_{k-1} = \{ z_0, \ldots, z_{k-1} \}$ is cosntructed; we take eacha $z_i \in Z_{k-1}$ and images of the inverse function f_c are calculated, this is: $f_c^{-1}(z_i)$. We repeat the process until a great amount of numbers are calculated and we draw them.

On the other hand, we can define the Julia set based on the border of the area of attraction of a fixed point z.

Comment 6

If $z \in C$ is an attractive fixed point; the attraction área of z is defined as:

$$A(z) = \left\{ y \in X \text{ t.q } \lim f^n(y) = z \right\} \qquad (1.2.3)$$

If $y \in A(z)$ then it has the orbit of y. If $z \in C$ is a periodic point of period $n > 1$, its orbit is called cycle, given byr:

$$\left\{ z, f^1(z) = z, f^2(z) = z, f^3(z) = z, \ldots, f^n(z) = z \right\}$$

Then, the attraction area of the cycle is defined as:

$$A(z) = \bigcup_{i=0}^{n} A\left(f^i(z)\right) \qquad (1.2.5)$$

From this, the Julia set is defined as the frontier of the area of attraction of a fixed point z.

Comment 7

A point $z \in C$ is a frontier point of Ω if:

$1.- B_r(z) \cap \mathbf{\Omega} \neq \varnothing$

$2.- B_r(z) \cap (C / \mathbf{\Omega}) \neq \varnothing$

Definition 8

The Fatou set is given by $C / J_c(f)$, i e, is the complement of the Julia set in C.

Theorem 2

The complement of a closed set is open.

This theorem does not say that the Fatou set is the maximum open such that the orbit of points converges, therefore has been dimensioned.

In other words, Julia and Fatuo observed that in certain cases, according the the values of c, the orbit of the points around the origin of a unit circle, converge in a fixed point of the function f_c, and are part of the attractionarea of c, while the orbit of the points most distanced from the origin tend to infinity.

Each of these two types of points constitutes a region, and in the middle remains what is called border, which is "infinitely thin", known as Julia set. In short, Julia sets are constituted by those periodic points of order $n > 1$ so its orbit is delimited and because they are the frontier of the attraction. Julia sets have a fractal structure that with the aid of computers have become visibles. What is more, Julia and Fatou proved that Julia sets associated to trasformations f_c, for any complex number c, could be of two types: connected or disconnected.

Comment 9

If $\mathbf{\Omega} \neq \varnothing$, $\mathbf{\Omega}$ is connected if and only if there are two open sets $\mathbf{A}$ and $\mathbf{B}$ then:

$1.- \mathbf{\Omega} \subset (\mathbf{A} \cup \mathbf{B})$

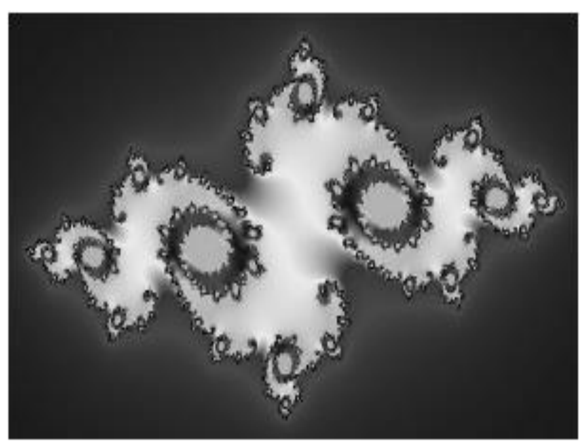
$2.- \mathbf{A} \cap \mathbf{\Omega} \neq \varnothing$ ó $\mathbf{B} \cap \mathbf{\Omega} \neq \varnothing$

$3.- (\mathbf{A} \cap \mathbf{\Omega}) \cap (\mathbf{B} \cap \mathbf{\Omega}) = \varnothing$

A result proved by Julia,in which he states that the Julia set associated with f_c is connected, according to the orbit of a fixed point $z_0 = 0$, with a value c arbitrary do not diverge or,is not connected if it does. This divergence lies totally on the norm of the complex constant c, as specified ahead:

Mandelbrot graphically represented the mathemathical work of Julia and Fatou; and he specifically focused in a subset of Julia sets, the connected Julia sets (see graphic 4).

Examples of not connected Julia sets (up), connected Julia set (bottom)



Graphic 4

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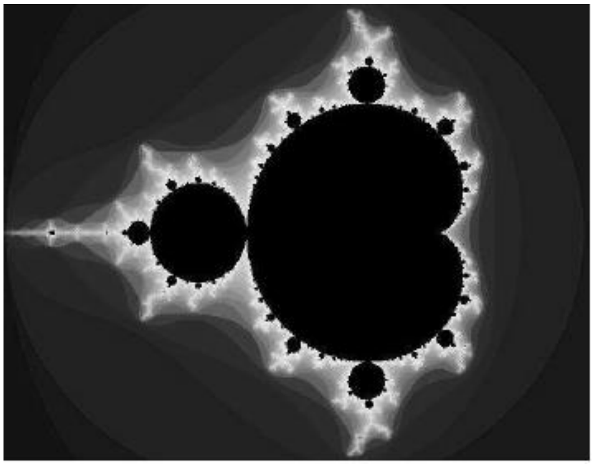

Fractal models in the financial market

The phenomenon of financial market have complex and dynamic nature, because the varibles composing it have a behaviour with self-similarity, brownian movements, non linearity, scale invariance, etc., so its is necessary to study them under the complexity theory; for example, financial market statistics have a behavior simikar to fractals (self similarity and similarity). The complex market dynamic is characterized by collective anomalus fluctuations that keep in mind the memories of critical phenomena, for example, the rate fluctuations do not merely follow a random process and the volatility of profitability is adjusted to autocorrelations in the long term, then we can say that the financial market is not a totally deterministic system nor totally stochastic but a combination of both, ie, it can be described by a series of equations nor through the theory of independent increments, nor identical distributions for different time intervals.

Therefore, it is required to create analysis techniques different from the traditional ones, since many of these involve linearity, seasonality, cause-effect relationships, among others, which makes them restrictive models that yield predictions unfaithful and / or imprecise, and then it is convenient to create new models that fit the reality of the financial data. Following this argument, Mandelbrot emphasized the description of the behavior of financial series, arguing the following four points:

- The pronounced changes in certain asset prices are much more frequent than the predicted by the Gaussian model, reflecting a leptokurtic character in the prices, this is, the probability of events associated with the tails of the distribution, is higher than what is assumed in theory.

- Great instant price changes occur very often, and give the impression that they must be explained causally as they do not conform to a stochastic prediction.

- The financial time series do not seem stationary, and the variance differs in various time periods.

- Changes in prices are not independent, showing diversity of patterns that provide some bases to technical analysis.

Under these four points, Mandelbrot assembled the fractal theory over the time series of stock returns (asset prices, financial rates, etc), since any kind of time series can be generated by a stochastic process and the information set observed is considered as a particular performance of the underlying stochastic process.

The fractal hypothesis (FH) of a financial series implies self-similarity; if accepted (FH true), it is said that economic cycles have long-term memory and this is much higher than the detected by autoregressive models, in which the process at a time h depends on their own values observed at previous periods.

R/S Analysis

The reescaled range analysis or R/S analysis is an estatistic method used to evaluate the occurrence of rare events, so it is used to describe financial shocks and collapses. The performance of a R/S analysis does not have to be limited to rare events, but rather be applied to any time series.

The result of a R/S analysis is the Hurst coefficient (also named fractal exponent), expressed by H, which is an indicator to determine if a time series has fractal behavior and, it measures the intensity of dependency in a long term of a time series.

The values H are found between 0 and 1, for which:

- $0 < H < .5$:= there is a negative correlation in the increseases to the time t
- $H = .5$ Is the Brownian movement, in which the increases are in dependent and, therefore, of correlation zero.
- $.5 < H < 1$:= there is a positive correlation between the increases, ie, if the graphic of X_t increases for a time t, then i t tends to keep increasing for $t' > t$.

It is noteworthy that there are softwares[43] able to calculate Hurst coefficient for any time series through different methods. In a time series with fractal structure is expected that the Hurst coeffiecnts calculatred by different methods[44] are similar.

Another field of study of Mandelbrot was the relation between his own fractal theory and the chaos theory, which is applied under circunstances where the process are random and the systems are dynamic, for this reason is widely used in economics and financial markets. Once discovered the Mandelbrot sets graphics, the role of fractals within the chaos theory could habe been seen, since both are iterative and dynamic systems; we can say that a fractal implies chaos, but a chaotic system not always generates fractal figures, in fact fractals seem to be a weird attractor in chaotic systems. Generally, chaos theory studies comples dynamic systems, within a framework of nonlinear relationships and presenting “sensibility to initial conditions” withouth existing divergence.

Chaotic systems are characterized by:

- Non-linear systems: a necessary condition is that the relationships that govern the system are non-linear; otherwise, it is impossible to generate endogenous dynamics aperiodic.

[43] El coeficiente de Hurst, puede ser calculado mediante el Programa H, implementado en lenguaje pascal, puede ser descargado de la siguiente dirección electrónica: http://www.bi.upv.es/~algarsal/hurst/hurst.zip; Otros softwares que tienen como utilería su cálculo, son Visual Chart, C++, entre otros.

[44] Los modelos mayormente usados para calcular el coeficiente de Hurst son: Crecimiento del Rango, Crecimiento del momento de orden dos y Momento de orden dos local.

- Perturbed systems: given the induction of a disturbance, the system reacts by changing its trajectory and evolving differently from its original state, ie the pre-disturbance state.

- They have sensitivity to initial conditions: Given an infinitesimal change in the values of the initial parameters, the evolution of the system diverges radically from its original state. Furthermore, the divergence is so large that makes it impossible to predict the final state of the system.
- They have attractor dynamics: while the system evolves in totally irregular and unpredictable form, it does not diverge, then long-term evolution remains bounded within a given subspace.

Chaos theory allows studying unstable systems with high volatility, divergences in their evolution, abrupt changes and no periodic cycles. Therefore, its main field of application in economics have been the stock markets.

Final considerations

The idea is to accept uncertainty as an unavoidable and decisive component of the financial marketsthus learn to live with it is the basic need to cover in the financial models, rather than minimize or ignore them as traditional models do. The question is "what to do" or "what research" in a chaotic system, and deepen in the study of Chaos Theory and Mandelbrot fractals is a priority; then the research should focus on the study of the uncertainty surrounding economic phenomena.

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