Capítulo 5 Familias normales, Teorema Grande de Picard y algunas de sus consecuencias para funciones analíticas

Chapter 5 Normal families, Picard Great Theorem and some of their consequences for analytic functions

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Resumen

El presente trabajo expositivo busca familiarizar al lector con la teoría, bien conocida, sobre la convergencia de familias de funciones analíticas, el estudio de las familias normales, teoría que aparece en la mayoría de los textos clásicos de Análisis Complejo. El objetivo de este compendio es presentar un capítulo con conocimientos básicos en Topología y Análisis Complejo, como herramientas, para explorar algunos resultados importantes relacionados con la teoría de Montel y el Teorema Grande de Picard. El trabajo contiene algunas consecuencias de los teoremas antes mencionados y varios ejemplos de sus aplicaciones, los cuales contribuyen a fortalecer en los lectores los antecedentes para el estudio de la Dinámica Holomorfa.

Teorema de Picard, Teoría de Montel, Funciones Analíticas

Abstract

The present expository work sought to familiarize the reader with a well-known theory of the convergence of families of analytic functions, the study of normal families, theory that appears in most of the classical texts in Complex Analysis. The objective of this compendium is to present a chapter with basic knowledge on Topology and Complex Analysis, as tools, in order to explore some important results related to Montel´s Theory and Picard´s Great Theorem. The work contains some of their consequences for analytic functions and several examples of their applications, which contributes to strengthen in the readers the background to study Holomorphic Dynamics.

Picard's Great Theorem, Montel's theory, Analytic Functions

1. Introduction

Given a transcendental entire function f in the complex plane it is possible to study the composition of f n- times i.e., $f^{n} = f \circ f^{n-1}$, where infinity for these functions is the only essential singularity. The notion of normal families forms a central feature of iteration theory of transcendental entire functions, because it helps to compare the orbits $f^{n}(z)$ of a given function f, under the iteration of f for different points z in the complex plane. In this expository document the reader will be able to identify the important results needed before going into the study of the iteration theory of transcendental entire functions. We deal with normal families and some interesting examples of it, also we give some results such as the Fundamental Normality Test which is essential to prove the Picard Great Theorem for analytic functions in a punctured disc around an essential singularity. We compare the Picard Great Theorem with Casorati-Weierestrass Property to see their differences. Also, we solve some problems to see how the results mentioned before are used in holomorphic dynamics.

In Section 2 we state the Selection Theorem, the Vitali´s Theorem and the Hurwitz´ Theorem. Also, we introduce the concept of normal family for analytic functions and some examples of it. The main theorem in Section 2 is The Fundamental Normality Test. In Section 3 we state the Picard's Great Theorem and some consequences for analytic functions. Also, the Casorati-Weieretsrass Property is given. Finally, in Section 4 we mention some examples of these two theorems to the theory of holomorphic dynamics.

2. Normal Families

In 1907 Paul Montel (1876-1975), a French a Mathematician, received his doctorate in Paris which was related to infinite sequences of both real and complex functions, A few years later Montel worked in complex function theory and introduced the theory of normal families, see [6] and [7]. Later his theory was extended to other kind of analytic functions.

Normal families may be understood from different points of view, for instance hyperbolic geometry, Kleinian groups, functional analysis and holomorphic dynamics among others.

We start by stating some well-known definitions from complex analysis which will be useful to define the concept of normal family. We refer to the reader to [1], [4], [5], [8].

I. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in a domain $G \subset \mathbb{C}$ *converges uniformly* to a function f if for all $\epsilon > 0$, there exists N such that if $n>N$, then $|f_n(z) - f(z)| < \epsilon$.

II. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in a domain $G \subset \mathbb{C}$ *converges locally uniformly* to a function f if it converges uniformly in each compact subset $K \subset G$.

III. A set F of functions is *locally uniformly bounded* in a domain G if for each compact subset $K \subset G$, there is a constant $M(K) < \infty$ such that for all $f\in F$ and for all $z \in K$, $|f(z)| \leq M(K)$, this is, F is uniformly bounded on K.

The following three theorems will be useful to determinate whether a family of functions is either normal or not. We state them without a proof, but they can be revised in [1]. We recall that an holomorphic function is an analytic function.

- ***Theorem 2.1. (Selection Theorem)*** *Let F be a set of functions which are all analytic in a domain G. If F is locally uniformly bounded in G, then there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset F$ which is locally uniformly convergent in G.*

Observe that the limit is analytic in G. The theorem also holds for uniformly bounded F where the uniform convergence is local.

- ***Theorem 2.2. (Vitali)*** *Let F be a set of analytic functions which is locally uniformly bounded in a domain G. If $\{f_n\}_{n\in\mathbb{N}} \subset F$ and $\{h_p\}_{p\in\mathbb{N}} \subset G$ and such that $lim_{n\to\infty}\, h_p$ exists and is equal to α with $\alpha \in G$, and if for every fixed p it is satisfied that $lim_{n\to\infty}\, f(h_p)$ exists, then the whole sequence $\{f_n\}_{n\in\mathbb{N}}$ is locally uniformly convergent in G.*

- ***Theorem 2.3. (Hurwitz's Theorem)*** *If a sequence $\{f_n\}_{n\in\mathbb{N}}$ of analytic functions converges locally uniformly to f in a domain G, and if for all $z \in G$ and $f_n(z)$ is not zero, then either f is never 0 or $f = 0$ for all z in G.*

The theorems above tell us about the properties that a limit function possess when it is considered on the functional space $\aleph$ the uniform convergence. Now, we proceed to define the concept of normal family, which determinate the characteristics of the compact sets in $\aleph$.

Normal Family. A family of analytic functions F on a domain $G \subset \mathbb{C}$ is *normal* in G if every sequence $\{f_n\}_{n\in\mathbb{N}} \subset F$ contains either (a) a subsequence converging uniformly to an analytic function $f = \infty$ on every compact subset $K \subset G$, or (b) a sequence converging uniformly to ∞ on every compact subset $K \subset G$.

The case (b) can also be expressed as follows: for all $M > 0$, there exists N such that for all $n > M$, $|f(z)| > M$, for all $z \in K$. An important result related to normal families is given as follows.

- ***Theorem 2.4.*** *A family F of analytic functions is normal in a domain G if and only if F is normal at every point $z_0 \in G$ (this is in some neighborhood of z_0).*

Proof. The necessary condition is clear, so we will prove only the sufficiency condition. Choose a countable dense subset $\{z_n\}$ of G where $z_n = x_n + iy_n$ and $x_n, y_n \in \mathbb{Q}$.

Let $D(z_n, r_n)$ be the largest disc about z_n in which F is normal. Also let K_n be an exhaustion of G by the compact sets $K_n = \{x \in G : |\, z\, | \leq n \text{ and } d(z, \partial G) \geq 1/n\}$. As z_n is dense in G it is covered by all $D(z_n, r_n/2)$, where F is normal in the disc $D(z_n, r_n/2)$. So if we pick a finite sub-cover of each K_n we obtain a countable cover $\cup_{n=1}^{\infty} D(z_n, r_n/2)$ of G.

For any sequence, one can extract a convergent subsequence $\{f_{n_k}\}$ which converges uniformly in the disc converges uniformly in the $D(z_n, r_n/2)$ either to an analytic function f or to ∞. The sequence has in a subsequence $\{f_{n_k}^{(1)}\}$ which converges uniformly in both $D(z_1, r_1/2)$ and $D(z_2, r_2/2)$.

Inductively, we get the diagonal sequence $\left\{f_{n_k}^{(k)}\right\}$ which converges uniformly in D(z_n, r_n /2) either to an analytic function or to ∞.

The distinction between ∞ and analytic functions splits the points $z \in G$, into two disjoint classes, say G_f and G_∞both are open and together form G, i.e., $G = G_f \cup G_\infty$. As G is a domain it is connected, so either $G_f = G$ or $G_\infty = G$.

Now, if K is a compact subset of G, then we can find a finite open cover, say $\cup_{n=1}^{j}$D(z_n, r_n /2) of K, where there is a subsequence converging either to an analytic function or to ∞. ■

From Theorem 2.1, we get that if a set F of analytic functions in a domain G is locally uniformly bounded in G, then F is a normal family. The following example illustrates it.

Example 1. If $F = \{f\colon f \text{ analytic in a domain } G\}$ is locally uniformly bounded in G, then so is $F' = \{f'\colon f \in F\}$ where f'denotes the derivate function of f.

Solution. By using Cauchy's formula for derivatives of an analytic function we have:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}, \tag{1}$$

Where C is a closed curve.

Where

$$|f^{(1)}(z)| = \left|\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\right| \leq \frac{1}{2\pi}\int_C \left|\frac{f(w)}{(w-z)^2}\right| |dw|. \tag{2}$$

Since f is locally uniformly bounded, we have that for all compact K = D(α, R) ⊂ G, there exists an M (K) < ∞ such that for all $f \in F$ and for all z ∈ K, | f(z)| ≤ M (K).
So for all z ∈ D(α, R/2) and for f ∈ F we have:

$$|f^{(1)}(z)| \leq \frac{1}{2\pi}\int_C \left|\frac{f(w)}{(w-z)^2}\right| |dw| \leq \frac{1}{2\pi}\frac{4M(K)}{R^2} 2\pi R = \frac{4M(K)}{R}. \tag{3}$$

Hence F' is also locally uniformly bounded, therefore a normal family. ■

Observation. Ahlfors in [1] mentioned that it not true that the derivatives of a normal family form a normal family. Consider the family of functions $f_n(z) = n(z^2 - n)$ in the whole plane. This family is normal, for it is clear that $f_n(z) \to \infty$ uniformly on every compact set. Nevertheless, the derivatives $f_n^{(1)}(z) = 2nz$ do not form a normal family, since they satisfy that $f_n^{(1)}(z) \to \infty$ if $z \neq 0$ and $f_n^{(1)}(z) \to 0$ for $z = 0$.

In what follows we consider an example which shows that normality can depend on the chosen domain.

Example 2. Let $F = \{f_n(z) = nz, n \in \mathbb{N}\}$. For any sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n(0) \to 0$ and if $z \neq 0$, $f_n(z) \to \infty$. So if we choose our domain G to be an open disc containing the origin, i.e., D(0,1), then F is not normal in this domain. But, if we consider the domain $G = \{z \in \mathbb{C} : |z| > 1\}$, then every sequence $\{f_n\}_{n\in\mathbb{N}}$ contains a subsequence converging uniformly to ∞ on every compact subset of G. ■

The following example shows that a family of analytic functions is normal if it omits an open subset of ℂ.

Example 3. If $= \{f\colon f$ is analytic in a domain $G \subset \mathbb{C}$ such that $|f(z) - a| > m > 0$, for some fixed $a \in \mathbb{C}\}$, show that $\mathfrak{F}$ is normal in G .

Solution. We know that the family

$$H = \left\{h\colon h(z) = \frac{1}{f(z)-a} \text{ with } f \in F\right\} \tag{4}$$

Is normal in G, since $|h(z)| < \frac{1}{m}$, for $z \in G$.

We must show that $\mathfrak{F}$ is normal in G. Consider the sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{F}$ and its corresponding sequence $\{h_n\}_{n\in\mathbb{N}} \subset H$which has a subsequence $\{h_{n_k}\}$. This subsequence $\{h_{n_k}\}$ has a corresponding $\{f_{n_k}\}$ (we do not know anything about it yet). By using Hurwitz's Theorem, we have that z ∈ G and so either (i) h(z) is not identically zero in G or (ii) h(z) = 0 in G.

Case (i). h(z) is not zero in G. Since $f(z) = a + \frac{1}{h(z)}$ we get that:

$$f(z) - f_{n_k}(z) = \frac{1}{h(z)} - \frac{1}{h_{n_k}(z)} = \frac{h_{n_k}(z) - h(z)}{h(z)h_{n_k}(z)}. \tag{5}$$

Let $K \subset G$ be compact. As $h_{n_k} \to h$ there exists for all $\epsilon > 0$ some natural number N so that for all $n_k > N$ and $z \in K$, $|h_{n_k}(z) - h(z)| < \epsilon$ and thus we obtain that

$$|f(z) - f_{n_k}(z)| = \frac{\epsilon}{\alpha(\alpha-\epsilon)}, \tag{6}$$

For $n_k > N$ and z ∈ K. Thus, for z in G $h(z) \neq 0$, so we get that $\mathfrak{F}$ is normal in G.

Case (ii). Consider $h(z) = 0$ in G, then $|h_{n_k}| < \epsilon$ for sufficiently large n_k (depends of ϵ) and z ∈ K, where K is a compact subset of G. Hence, for ϵ small enough. Thus $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{F}$ converges uniformly to ∞ on K, thus we have that $\mathfrak{F}$ is normal in G.■

The following result gives a sharper result than Example 3, in the literature it is known as the Montel's three omitted values theorem.

- ***Theorem 2.5. (Fundamental Normality Test)*** *Let $\mathfrak{F}$ be a family of analytic functions on a domain G ⊂ $\mathbb{C}$ which omits two fixed values a and b in $\mathbb{C}$. Then $\mathfrak{F}$ is normal in G.*

Proof. Assume without loss of generality that a = 0 and b = 1. Fix some $z_0 \in G$and pick ρ>0 so that $D(z_0, \rho) \subset G$. By considering the rescaled function $g(z) = f(z_0 + \rho z)$ in D(0,1) we obtain by using the sharp form of Schottky, see [8], so we obtain that

$$|f(z)| \leq |\alpha| \leq R = 1. \tag{7}$$

So, there exists a bound M (1, 1/2) > 0 such that $|f(z)| \geq \left|M\left(1, \frac{1}{2}\right)\right|$ for all ∈ $\overline{D}\left(z_0, \frac{\rho}{2}\right)$. On the other hand, if $|f(z_0)| > 1$, then $\left|\frac{1}{f(z)}\right| \leq \frac{1}{M\left(\frac{1,1}{2}\right)}$ for all $z \in \overline{D}\left(z_0, \frac{\rho}{2}\right)$ since $\frac{1}{f(z)}$ omits 0 and 1. Thus, $\mathfrak{F}$ can be written as follows:

$$\mathfrak{F} = F_1 \cup F_2 = \{f \in \mathfrak{F}\colon |f(z_0)| \leq 1\} \cup \{f \in \mathfrak{F}\colon |f(z_0)| > 1\}, \tag{8}$$

Where F_1 is normal in $\overline{D}\left(z_0, \frac{\rho}{2}\right)$ by the Selection Theorem and the definition of normal family F_2 is normal by the same argument. Now, for any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{F}$ we can choose a subsequence from either F_1 or F_2 such subsequence has in turn a locally uniformly convergent sequence, thus $\mathfrak{F}$ is normal in $D\left(z_0, \frac{\rho}{2}\right)$.

Since this hold for any $z_0 \in G$, then $\mathfrak{F}$ is normal in G by Theorem 2.4.■

Theorem 2.5 in connection with Example 3 gives the following corollary.

- ***Corollary 2.6***. *A family of analytic functions on some domain $G \subset \mathbb{C}$ which is not normal omits at most one finite point.*

3. Picard´s Great Theorem and Some Consequences for Analytic Functions

In this section we proceed to give some definitions of singularities of analytic functions and then we state the Picard´s Great Theorem.

If f is not analytic in any disc D (a, r), where r > 0, we say that a is an *isolated singularity* of f, also a is called a *singular point*. For functions we study the characteristics of a function on a neighborhood of a singular point by using Laurent's series.

Briefly speaking a Laurent's series is a double series of the form $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ where

$$c_{-n} = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_o)^{-n+1}}; \tag{9}$$

And

$$c_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_o)^{-n+1}}. \tag{10}$$

Using Laurent's series, we can classify a singular point a in terms of the quantity of the coefficients c_{-n}, which can be an arduous labor for some functions. Thus, the following statements simplifies the situation:

1. If f is bounded in some neighbourhood D (a, r) of a. In this case a is called *a removable singularity*.

2. If f (z) → ∞ as z → ∞, a is called a *pole*.

3. If f is unbounded and does not converge to ∞ for z → ∞. In this case a is called an *essential singularity*.

Now, let f be analytic in the punctured disc D (0, R) with the Laurent's expansion. Let f be an analytic function in the complex plane or the Riemann sphere. We say that w is an *omitted value* of f, if for every z we have f (z) − w = 0 has no solutions and w is a Picard exceptional value of f, if f (z) − w = 0 has finite solutions.

Observe that if w is an omitted value of f, then w is a Picard exceptional value of f.

Example 4. (i) The exponential map e^z has two omitted values, zero and ∞ which are also Picard exceptional values.

(ii) The map e^z/z has zero as an omitted value, so it is a Picard exceptional value, while ∞ is as a Picard exceptional value but it is not an omitted value. ■

From the above example in mind we can state the following result.

- ***Theorem 3.1. (Picard Great Theorem)*** *Let f be analytic in the punctured disc D (z_0, R) where z_0 is an essential singularity of f. Then f attains every finite complex value, infinitely often in D (z_0, R), with at most one possible exception.*

Proof. Assume that $D = \{0 < |z| < R\}$ and suppose that there are two values a and b which are omitted by f in D. The family of functions $\mathfrak{F}$ defined by $f_n(z) = f\left(\frac{z}{2^n}\right), n \in \mathbb{N}$ are analytic in the annulus $A = \left\{\frac{R}{2} < |z| < R\right\}$ and do not assume the values a or b in A. Since $\mathfrak{F}$ is normal in A, there is a subsequence $\{f_{n_k}\}$ converging uniformly to $\hat{f}$ on the compact set $\left\{|z| = \rho: \frac{R}{2} < \rho < R\right\}$, where $\hat{f}$ is either an analytic function or equal to ∞ in A.

If $\hat{f}$ is analytic, then its boundedness on $|z| = \rho$ which implies that $\{f_{n_k}\}$ is uniformly bounded on $|z| = \rho$, say $|f_{n_k}(z)| \le M,\ |z| = \rho, where\ k \in \mathbb{N}$.

Then,

$$|f_{n_k}(z)| \le M,\ |z| = \frac{\rho}{2^n},\ \text{where } k \in \mathbb{N}. \tag{11}$$

That is, f is bounded on a sequence of concentric circles converging to the origin. By the Maximum Modulus Theorem, $|f(z)| \le M$ in the region between any two circles. Therefore,

$$|f_{n_k}(z)| \le M,\ 0 < |z| \le \frac{\rho}{2^n}, \tag{12}$$

Which contradicts the fact that f must be unbounded in any neighbourhood of an essential singularity.

If $\hat{f} = \infty$ in A, again we obtain a contradiction, see [7] for a proof. Finally, if there are two values, say α, β which are attained only finitely often by f, then in some sufficiently small deleted neighbourhood of the origin f would omit α, β and the result follows. ■

We recall that transcendental entire functions are analytic in ℂ and ∞ is an essential singularity. Some examples are the exponential and the sine functions. From Picard Great Theorem it can be deduced the following result by putting $z_0 = \infty$.

- ***Corollary 3.2.*** A transcendental entire function attains every finite complex value infinitely often, with at most one possible exception.

An application of the theorem above is the following example.

Example 5. Do there exist any non-constant entire functions f and g such that

$$e^{f(z)} + e^{g(z)} = 1 \text{ for all } z \in \mathbb{C}? \tag{13}$$

Solution. The answer is negative.

Indeed, we know that $e^{f(z)} f = 0 f= e^{g(z)}$ for any f (z) or g(z). So by Corollary 3.2 we have that $e^{f(z)}$ must again all other values in ℂ, for instance if $e^{f(z)} + e^{g(z)} = 1$ is fulfilled by some f and g, then $e^{f(z)} = 1 - e^{g(z)}$. But since $e^{g(z)} = 0$ for z ∈ ℂ, we obtain a contradiction.■

As we can see the example above is an immediate application of the Picard´s Great Theorem, moreover, we also in the same way can prove the following facts:

1. If a meromorphic function on ℂ misses three values, then it is constant.
2. If f is entire and one-to-one, then it is linear.
3. If f, g are entire functions and g'=(g), then f is linear or g is constant.
4. If X is a Riemann sphere with n punctures, then for $n \ge 3$ the universal covering space $\tilde{X}$ of X is the upper-half plane.

Based on the facts above and the definition of essential singularity it led us to wonder about the characteristics that such functions possess. Casorati-Weierestrass proved independently the following property which established a fascinating behaviour of the essential singularity, see proof in [8].

- ***Theorem 3.3. (Casorati-Weierestrass Property)*** Let f be analytic in the punctured disc $D(a, R)$, where a is an essential singularity of f. Then for all w ∈ ℂ, and any constants $s > 0$, R > 0, we have $|f(z) - w| < s$ for some $z \in D(a, R)$.

Observation: The Great Picard Theorem is a substantial strengthening of the Casorati-Weierestrass Property which describes the behaviour of a holomorphic functions near their essential singularities.

Example 6. Observe that in the above theorem it is not claimed that f attains the value $w \in \mathbb{C}$, that is, $f(z) = w$ has at most one solution, i.e., f is univalent. Show that $f(z) = az + b$, with a, b constants.

Solution. Assume that $f \neq c$ for some $c \in \mathbb{C}$.

Then by the Open Mapping Theorem, the image of an open disc D (0,1) is open under f (as f is analytic there). So certainly, D ($f(0)$, ρ) ⊂f (D (0, 1)) for some ρ > 0. Then for all z not in the disc D (0, 1) the function f never lie in the disc D(f (0), ρ). Thus, in the neighbourhood $1 > |z| > \infty$ of ∞, we have $| f(z) - f(0) | \geq \rho$. Thus, by the converse of Theorem 3.3, ∞ is not an essential singularity of f. Our definitions on singularities hence tells us that the Laurent's expansion of f must be of the form $\sum_{n=0}^{\infty} a_n z^n$ since f is entire. Thus, we have a polynomial, and since it is univalent, it must be of degree one. Therefore, $f(z) = az + b$ with a, b constants. ■

4. Applications in Holomorphic Dynamics

Montel's theory of normal families is quite important in the iteration of analytic functions. Between 1918 and 1920, two French mathematicians, Pierre Fatou (1878-1929) and Gaston Julia (1893-1978) obtained several results related to the iteration of rational functions of a single complex variable. Each of them based his approach on Montel's theory of normal families. The main objects of the theory are the maximal domains of normality and its complement. In this section we will give some basic facts in iteration theory using some results of the previous sections.

Given a transcendental entire function f, it is possible to study the sequence formed by its iterates denoted by $f^n := f \circ f^{n-1}, f^0 = Id$ and $n \in \mathbb{N}$, this is, the composition of f with itself n-times. For this class of functions infinity is the only essential singularity.

The following example shows how the results in Section 3 can be applied to iteration theory.

Example 7. If f and g are transcendental entire functions, then $f \circ g = f(g)$ is a transcendental entire function.

Solution. Take $w = g(z)$ and look at $f(w)=f(g(z))=v$, where v is not a Picard value of f. Then by Theorem 3.1, there exists infinitely many solutions $w_1.w_2, \dots, w_n, \dots$ for this equation. Therefore, the equations $g(z)=w_1$, $g(z_2) = w_2, \dots g(z) = w_n, \dots$ have infinitely many solutions, except for maybe one $g(z) = w_i$. But still, this yields only have finitely many solutions and the composition of two transcendental entire functions is also a transcendental entire, we have shown that $f \circ g$ is transcendental entire.

We can use Example 7 n-times to show that if f is a transcendental entire function, then f^n is also a transcendental entire function. ■

Given f a transcendental entire function we define orbits which are related to the iteration of an analytic function f.

The *forward orbit* of a point z is $O^+(z) = \{w: f^n(z) = w, \text{ for } n \in \mathbb{N}\}$.

The *backward orbit* of a point z is $O^-(z) = \{w: f^n(w) = z, \text{ for some } n \in \mathbb{N}\}$.

The *grand orbit* of a point z is $O(z) = O^+(z) \cup O^-(z)$.

Also, we define $E(f)$ as the set of Fatou exceptional values of f, this is, points whose inverse orbit $O^-(z)$ is finite.

Example 8. The map f (z) = e^z has Fatou exceptional values zero and ∞ (both are also omitted values, see Example 4).

If we add a pole in the function above the behaviour changes.

Example 9. The map h (z) = − e^z + 1/z, has no omitted values, the Picard exceptional value is ∞, but it is not a Fatou exceptional value since $f^{-n}(\infty)$. Thus $E(h) = \emptyset$.

With the definition of normal family, see Section 2, it is possible to investigate two sets in the complex plane giving a transcendental entire function f: The Stable set (Fatou set) $\mathfrak{F}(f)$ which is defined as the set of all points $z \in \mathbb{C}$ such that the sequence of iterates $\{f^n\}_{n\in\mathbb{N}}$ forms a normal family in some neighbourhood of z. The Chaotic set (Julia set), denoted by $\mathfrak{J}(f)$ is the complement of the Fatou set.

To prove some important properties of the two sets mentioned above, it is necessary the concept of normal family and some results of Complex Analysis which drives us to the subject of holomorphic dynamics. We recommend the reader to revised [2] and [3] for starting the subject in holomorphic dynamics for rational and transcendental entire functions.

5. Research Method

This paper presents two main results used in the iteration of holomorphic functions, based on classical textbooks used in basic courses of Complex Analysis, which open a window to the readers to enter in this magnificent and depth area of Complex Dynamics.

6. Results

Montel's criterion and Picard's Great Theorem are depth results of Complex Analysis which are equivalent. This equivalence is a bridge between compacity in the space of holomorphic functions and the behavior of a holomorphic function around an essential singularity. Moreover, the results stated in this chapter can be used as model in different areas such as Kleinian groups and iteration of holomorphic functions. In this document we try to explain some important concepts in a simplified an easier way to start the study the area of Holomorphic Dynamics.

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