Chapter 10 A study of the Apollonian Gasket

Capítulo 10 Un estudio del Tamiz de Apolonio

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Abstract

The present expository work sought to familiarize the reader with a well-known geometrical object obtained as the recursive application of the solution of the Apollonius’ problem known as the Apollonian gasket. This object appears in Geometry, but also in other branches of mathematics such as Continuum Topology and Kleinian Groups. The work contains some properties of this object, the statement and partial solution of the famous Apollonius’ problem

Apollonius’ Problem, Apollonian Gasket, Continuum

1. Introduction

In the third century B.C., Apollonius of Perga wrote two books on Contacts. Since no copy of Apollonius' Contacts has survived the ages, Pappus of Alexandria deserves credit for eternally linking Apollonius' name with the tangents problem.

The tangency problem is well known, however, and unlike many of the classical Greek geometric construction problems, this one has a solution: given three geometric objects, each of which may be either a point, a line, or a circle, under what conditions is possible to construct a circle which passes through each of the points and touches the given lines and circles.

Since the conditions of the problem allow for any combination of circles, lines, and points, this rises to ten possible cases. As Coxeter (1968) mentions Euclid's Elements already cover the most straightforward (three points and three lines). Apollonius treated these two cases together with these other six (two points and a line; two lines and a point; two points and a circle; two circles and a point, two circles and a line; a point, a line, and a circle) in Book I of the Tangences, and the two remaining cases (two straight lines and a circumference, and three circumferences) in Book II of the Tangences. Although unfortunately, these books were lost through Pappus of Alexandria (4th century A.D.). It is known that Apollonius solved the first nine, and today it is believed that Isaac Newton was the first mathematician who solved the problem of finding the circle tangent to three other circles through the rule and the compass.

A particular case of Apollonius' problem is known today as the three coins problem, or kissing coins problem. In this variant, the three circles of possibly different radii are taken to be mutually tangent. There are two solutions to this particular case of Apollonius' problem: a small circle where all three given circles are externally tangent and a large circle where the three given circles are internally tangent. In 1643 Renè Descartes sent a letter to Princess Elisabeth of Bohemia in which he provided a solution to this particular case of Apollonius' problem, and his solution became known as Descartes' circle theorem.

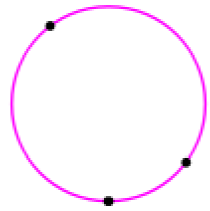
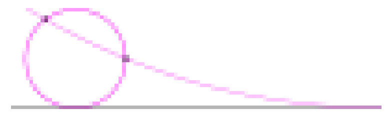
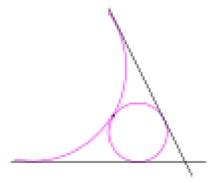
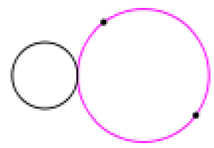
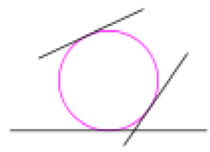
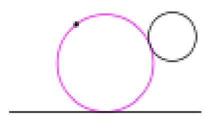
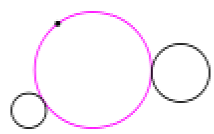
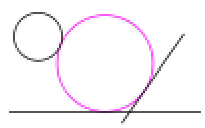
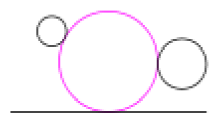
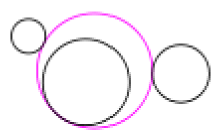
In the present expository work, we study the solution to Apollonius' problem with different Geometry approaches such as the Euclidean, Analytic, Conformal and Geometry of the invariants. Section 1 deals with the solution to Apollonius' problem.

In section 2, we study the famous Descartes' four-circle theorem. In section 3, we explore the existence of the apollonian Gasket through Conformal Geometry and finally, in Section 4, we study the Apollonian Gasket through the consistent geometry approach.

2. Solution to Apollonius' Problem

Apollonius’ problem requires to construct one or more circles tangent to three given objects in a plane, which may be either circles, points, or lines. This gives rise to ten types of Apollonius' problem, one corresponding to each combination of circles, lines, and points, which may be labeled with three letters, either **C**, **L**, or **P**, to denote whether the given elements are a circle, line, or point, respectively (see Table 1). As an example, the type of Apollonius problem with a given circle, line, and point is denoted as **CLP**.

Table 1: Ten Types of Apollonius' Problem

Case	Code	Given Elements	Solution	Example (solution in pink; given objects in black)
1	**PPP**	three points	The problem is reduced to constructing a circle that contains them	
2	**LPP**	one line and two points	The problem is reduced to constructing a circle tangent to the line and containing the two points	
3	**LLP**	two lines and a point	The problem is reduced to constructing a circle that is tangent to the two lines and contains the point	
4	**CPP**	one circle and two points	The problem is reduced to constructing a circle that contains the two points and is tangent to the given circle	
5	**LLL**	three lines	The problem is reduced to constructing a circle that is tangent to the three lines	
6	**CLP**	one circle, one line, and a point	The problem is reduced to constructing a circle that passes through the point and is tangent to the line and the given circle	
7	**CCP**	two circles and a point	The problem is reduced to constructing a circle that contains the point and is tangent to the two given circles	
8	**CLL**	one circle and two lines	The problem is reduced to constructing a circle that is tangent to the lines and to the given circle	
9	**CCL**	two circles and a line	The problem is reduced to constructing a circle that is tangent to the line and to the two given circles	
10	**CCC**	three circles (the classic problem)	Construct a circle that is tangent to three given circles	

For the solution of the problem we need to consider the configuration of the given objects, i.e., the position in which the objects are situated in the plane lo which arises to the number of solutions of the problem for each case, which makes this problem more interesting. In our case we will give the constructions for two cases: Cases 8 and 10 use rule and compass.

Case 8: Given two lines and a circle, we must construct a circle that is tangent to the lines and to the given circle.

Construction (step by step, see Figure 1):

Configuration 1: When the lines are parallel, and the circumference is tangent to the two lines.

Configuration 2: When the circumference is between two straight lines L and M

1. Determine the radius of the given circle O.

2. Draw parallels to each side of the line L, at a distance equal to the radius of the circumference.

3. Draw the bisector of the angle formed by the lines L and M.

4. Find the symmetrical point of the center O with respect to the bisector. (Point O').

5. Draw the line OO'.

6. Mark the intersection between the line OO' and the parallel line L1. (point M).

7. From M, draw tangents to the circle of diameter OO'. Mark the points of tangency D and E.

8. Draw the circle with center at M and radius MD.

9. Find the intersections of the parallel L1 with the circle CM with center M.

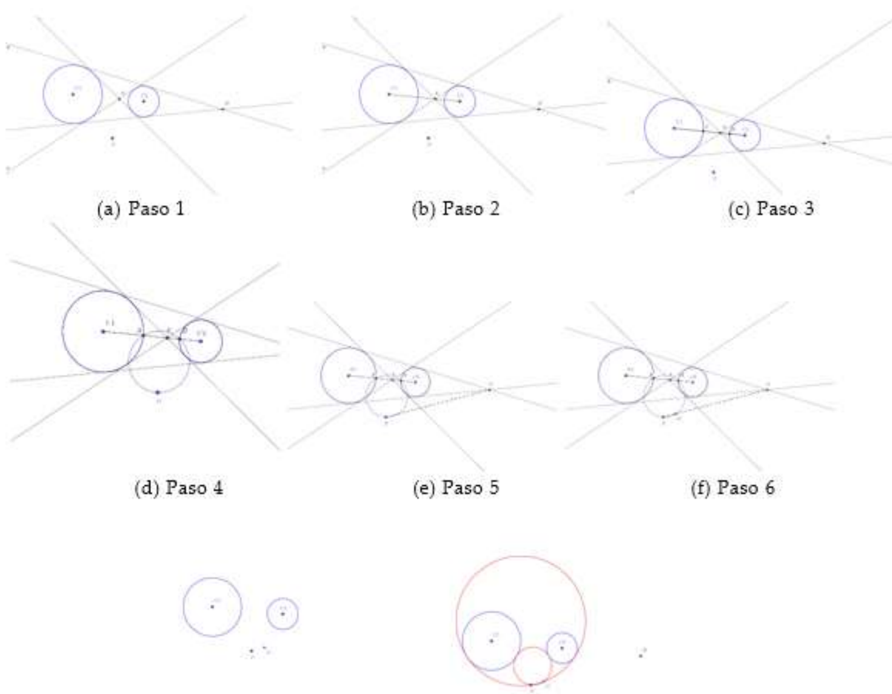
10. Draw the perpendiculars to the parallel L1 through points A and B.

11. Find the intersections of the perpendiculars found with the bisector.

12. The points P and Q are the centers of the circles sought. Draw circles with centers at P and Q tangent to lines L and M.

13. If these steps are followed for the parallel line L2, two other circles are obtained.

Figure 1 Case 8 Configuración 2



(a) Paso 1 (b) Paso 2 (c) Paso 3

(d) Paso 4 (e) Paso 5 (f) Paso 6

(g) Paso 7 (h) Pasos 8, 9

Case 10: Given three circles, we must construct a circle that is tangent to the three given circles.

Construction with steps (see Figure2):

1. Draw two auxiliary circles with center at O2 and O3 and radius r2 r1 and r3 r2 respectively. Aux1, Aux2.

2. Unite the centers O2 and O3 and find their intersection with the circles Aux1, Aux2. (R, S).

3. Draw a segment perpendicular to center O2 and center O3.

4. Find the intersections of the segments with their respective auxiliary circles.

5. Join the intersections, find the point of intersection (M) this will be the center of inversion.

6. Join point M with center O1 (MO1).

7. Join center O1 with points R and S.

8. Draw the circumference that passes through R, S, O1 (O4).

9. Find the point of intersection of O4 with the line MO1 (O1′).

10. Draw a line that passes through the points of intersection of the circumference O4 and the circumference Aux2.

11. Find the intersection of the previous line with line MO1.

12. Draw the PO2 segment.

13. Find the midpoint of the segment PO2 (h).

14. Draw an arc of circumference with center at h and radius hO2.

15. Find the intersection points of the arc with the circumference aux2 (T 1, T 2).

16. We draw the perpendicular bisector of the segment O1O1′.

17. Join the points T 1, T 2 with the point O2 and find the points of intersection with the perpendicular bisector of O1O1′ (O5, O6).

18. Join the points O5, O6 with the initially given centers O1, O2, O3 and find the points of intersection T 3, T 4, T 5, T 6, T 7, T 8 (points of tangency).

19. Draw the circles with center at O5, O6 and radius towards one of their tangency points.

20. Those circles are the solutions to the problem.

Figure 2 Steps for Case 10

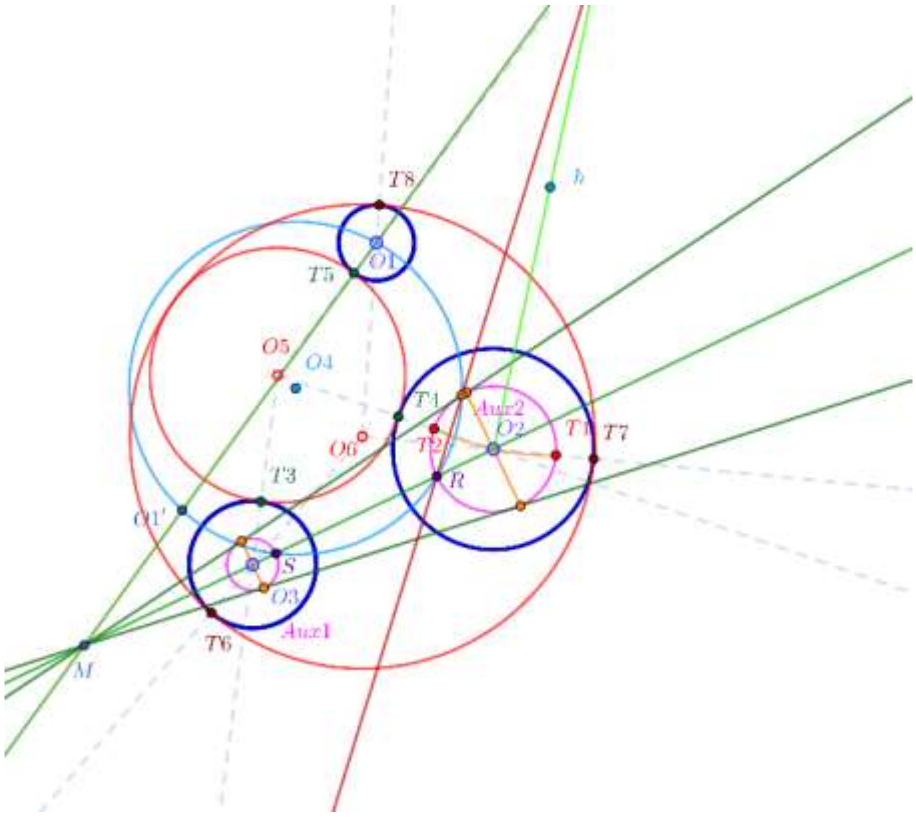
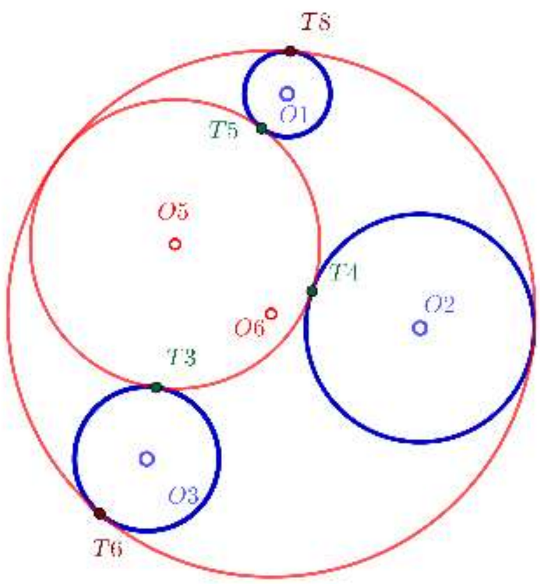


Figure 3 Solution for a configuration for Case 10



3. Descartes' circle theorem

In 1643, Descartes wrote to Princess Elizabeth of Bohemia (1618-1680) stating formula he had established on the radii of the tangent circles, and for which she independently provided a proof. The radii are related by the following formula.

Theorem 2.1 (Descartes-Princess Elizabeth). Assume that the radii of the original circles are $a, b, c, > 0$ and the fourth mutually tangent circle has radius $d > 0$, then

$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2.$$

Proof. Suppose four circles lying in a plane, such that any two of them touch each other externally (meaning that in each pair of touching circles each center is external to the other circle of the pair). If the centres of a pair are P, Q, then the point of tangency lies on the line PQ, and the length PQ is equal to the sum of the radii. So, if P, Q, R, S are the centers of the four circles, and a, b, c, d are their radii, we have the six following equations:

$$\boldsymbol{PQ = a + b} \quad \boldsymbol{QR = b + c} \quad \boldsymbol{RP = c + a}$$

$$\boldsymbol{PS = a + d} \quad \boldsymbol{QS = b + d} \quad \boldsymbol{RS = c + d}$$

Now, suppose that S is inside the triangle PQR and let

$$\theta = angle\ PSQ \quad \phi = angle\ QSR \quad \psi = angle\ RSP,$$

then

$$\theta + \phi + \psi = 2\pi$$

Applaying to the triangle PSQ the cosine rule we have

$$(a + b)^2 = (a + d)^2 + (b + d)^2 - 2(a + d)(b + d)\, cos(\theta)$$

Therefore

$$cos(theta) = \frac{2d^2+2ad+2bd-2ab}{2(a+b)(b+d)} = \frac{AB-2ab}{AB} = 1 - \frac{2ab}{AB}$$

where we have written A for a + d and B for b + d. Hence

$$s = sin(\theta/2) = \sqrt{\frac{1-cos(\theta)}{2}} = \sqrt{\frac{ab}{AB}} = \sqrt{\alpha\beta}$$

where we have written α for a/A and β for b/B. As before we obtain:

$$t = sin(\phi/2) = \sqrt{\frac{bc}{BC}} = \sqrt{\beta\gamma}$$

$$u = sin(\psi/2) = \sqrt{\frac{ca}{CA}} = \sqrt{\gamma\alpha}$$

where we have also set C = c + d and γ = c/C.

Now, since the angles θ/2, φ/2 and ψ/2 add up to π , this implies that

$$s = sin(\theta/2) = sin(\pi - \phi/2 - \psi/2) = sin\left(\frac{\phi+\psi}{2}\right) = t\sqrt{1-u^2} + u\sqrt{1-t^2}$$

This relates *s*, *t* and *u*, but the equation is difficult to follow. It can be improved by getting rid of the square roots. Thus, squaring both sides of the equation and simplifying we obtain that

$$s^4 + t^4 + u^4 - 2(s^2t^2 + t^2u^2) + 4s^2t^2u^2 = 0$$

This is much better, being nicely symmetrical between s, t and u. We can get that the equation becomes

$$2(s^4 + t^4 + u^4) - (s^2 + t^2 + u^2)^2 + 4s^2t^2u^2 = 0$$

Substituting the expressions for *s, t* and u derived from the cosine rule, we obtain

$$2(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) - (\alpha\beta + \beta\gamma + \gamma\alpha)^2 + 4\alpha^2\beta^2\gamma^2 = 0$$

Dividing through by $\alpha^2\beta^2\gamma^2$,

$$\left(\frac{1}{\gamma}+\frac{1}{\alpha}+\frac{1}{\beta}\right)^2 = 2\left(\frac{1}{\gamma^2}+\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)+4.$$

Now

$$\frac{1}{\alpha}=\frac{A}{a}=\frac{a+d}{a}=1+\frac{d}{a}$$
$$\frac{1}{\beta}=1+\frac{d}{b}$$
$$\frac{1}{\gamma}=1+\frac{d}{c};$$

and so, writing σ for $d/a + d/b + d/c$, τ for $(d/a)^2 + (d/b)^2 + (d/c)^2$, the equation becomes

$$(3+\sigma)^2 = 2(3+2\sigma+\tau)+4.$$

Therefore,

$$2\tau = -1+2\sigma+\sigma^2 = (\sigma+1)^2-2.$$

And

$$2(\tau+1) = (\sigma+1)^2$$

Thus,

$$2\left(1+\frac{d^2}{a^2}+\frac{d^2}{b^2}+\frac{d^2}{c^2}\right) = \left(1+\frac{d}{a}+\frac{d}{b}+\frac{d}{c}\right)^2.$$

Dividing through by d^2, we get

$$2\left(\frac{1}{d^2}+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2.$$

and we are done.

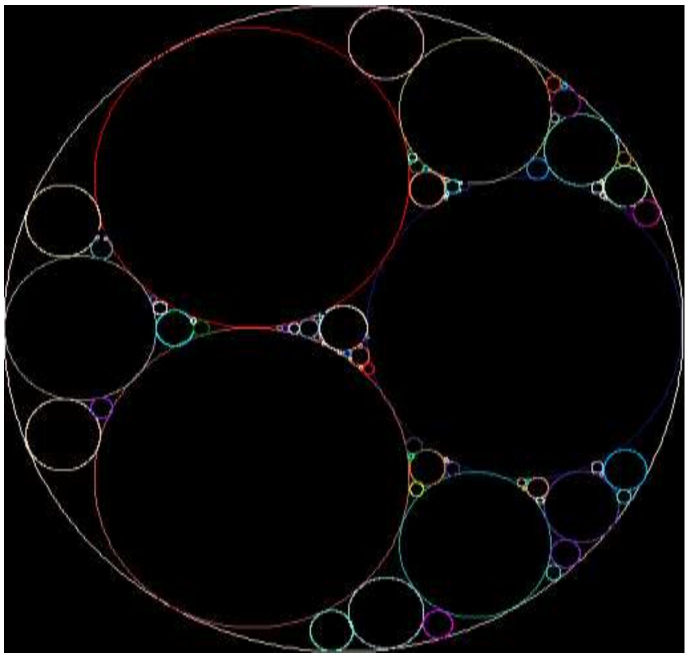
An immediate consequence of Descartes' circle theorem is the following: given the curvature of three mutually tangent circles, C1, C2 and C3 we can solve for the curvatures of the two circles that are mutually tangent to the three original circles. From this new collection of four mutually tangent circles, we can arbitrarily choose three of them and solve for the curvature of two new circles that are mutually tangent to this selection of three circles, these are called Apollonian circles. Adding the two Apollonian circles C4 and C5 to the original three, now we have five circles.

Take one of the two Apollonian circles, say C4. It is tangent to C1 and C2, so the triplet of circles C4, C1 and C2 has its own two Apollonian circles. We already know one of these, it is C3, but the other one is a new circle, say C6.

Similarly, we can construct another new circle C7 that is tangent to C4, C2 and C3, and another circle C8 from C4, C3 and C1. This gives us 3 new circles. We can build another three new circles from $C_5,$ giving six new circles in total. Together with the circles C1 to C5, it gives a total of 11 circles.

Continuing the construction stage by stage in this way, we can add 23n new circles at stage n. The sizes of the new circles are determined by Descartes' theorem, the limit set is called Apollonian Gasket or Apollonian packing as Leibniz named it, who was the first to carry out the construction, see Figure 3.

Figure 3 Apollonian Gasket



4. Apollonian Gasket and Möbius transformations

In this section we will construct the Apollonian gasket using Conformal Geometry through Möbius transformations.

A Möbius transformation of the complex plane is a rational function of the form

$$T(z) = \frac{az+b}{cz+d}.$$

With a, b, c, d complex numbers. Non-identity Möbius transformations are in generall classified into four types; parabolic, elliptic, hyperbolic and loxodromic, with the hyperbolic being a subclass of the loxodromic. The classification has both algebraic and geometric significance. The four types can be distinguished by looking at the trace, denoted by tr T= a+ d. In Table 2 there is a resume of the the general classification of the Möbius transformations.

Table 2 Classification of Möbius transformations

Transformation	Trace squared	Multipliers	Class representative	
Circular	$\sigma = 0$	$k = -1$	$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$	$z \mapsto -z$
Elliptic	$0 \le \sigma < 4$	$\|k\| = 1$	$\begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}$	$z \mapsto e^{i\theta}$
Parabolic	$\sigma = 4$	$k = 1$	$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$	$z \mapsto z + a$
Hyperbolic	$4 < \sigma < \infty$	$k \in R^+$ $k = e^{\pm\theta} \neq 1$	$\begin{bmatrix} e^{\theta/2} & 0 \\ 0 & e^{\theta/2} \end{bmatrix}$	$z \mapsto e^{\theta}z$
Loxodromic	$\sigma \in C \setminus [0,4]$	$\|k\| \neq 1$ $k = \lambda^2, \lambda^{-2}$	$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$	$z \mapsto kz$

Theorem 4.1. There exists a unique Apollonian Gasket.

Proof. Consider three circles with equal radii r and centers at cubic roots of unity: 1, ω,ω^2. Label the circles X_1, X_2, X_3 in counterclockwise order starting with the one whose center is at 1. We seek a fourth circle X_0 and three Möbius maps T_1, T_2, T_3 such that:

$T_i(X_j) = X_j, j \neq i, i, j = 1,2,3$

$T_i(X_i) = X_0$

$T_2 = RT_1R^{-1}$

$T_3 = R^{-1}TR,$

where R is a rotation of H around the origin. It follows that X_0 will have center at 0 because it is invariant under R.

One can construct T_1 as a composition: $T = SJ$, where S is the reflection through the axis of symmetry of $X_2$and $X_3$and J is the inversion with respect to a circle orthogonal to X_2 and X_3. Moreover,

$$T_1(z) = \frac{I-1}{2}\frac{2z-I}{z+I}.$$

The fixed points f and $f^*$of $T_1$are the points of intersection of the inversion circle J with the axis of symmetry of X_2 and X_3. Moreover, $f = -\frac{1}{2} + i\beta$ where $\beta = \sqrt{\frac{3}{4} - r^2} = \frac{1}{2}\frac{\sin\alpha}{\sqrt{\cos^2\alpha - 2/3}}$.

The angle α is defined by $\cos\alpha = \frac{1}{2}\sqrt{\frac{9-8r^2}{3(1-r^2)}}$.

The radius ρ_0 of X_0 is given by $\rho_0 = \frac{2r^2-3+\sqrt{9-8r^2}}{2r}$.

Consider $X_\infty \coloneqq T_1^{-1}(X_1)$ and since T_1 is Möbius transformation and X_1 is invariant under T_1, it follows that X_∞ is a circle. We can reduce that its radius is given by $\rho_\infty = \frac{\sqrt{\cos 2\alpha + 1\sqrt{3\cos 2\alpha - 1}} + \sqrt{3}\cos 2\alpha}{\sqrt{2(2\cos 2\alpha - 1)(3\cos 2\alpha - 1)}}$ and center 0. Therefore, $X_\infty = T_i^{-1}(X_i), i = 1,2,3$ and $T_1 = \begin{pmatrix} \frac{2(I-1)}{\sqrt{6I(I-1)}} & \frac{-I(I-1)}{\sqrt{6I(I-1)}} \\ \frac{2}{\sqrt{6I(I-1)}} & \frac{2I}{\sqrt{6I(I-1)}} \end{pmatrix}$ which is:

(i) parabolic if and only if $r^2 = \frac{3}{4}$;

(ii) elliptic if $r^2 \geq \frac{9}{8}$ o $r^2 < \frac{3}{4}$.

For further details of this affirnation the reader can consult Lagarias J., Mallows C. & Wilks A. (2002). Thus, when T_1 is a parabolic transformation, we obtain that X_0 and X_∞ are the desired circles. Furthermore, if we consider any other triad of circles, it is possible to construct a Möbius transformation that sends the three given circles into the given circles. Therefore, the Apollonius Gasket is unique.

5. Apollonian Gasket and Kleinian groups

To establish how the Apollonian Gasket is related to Kleinian Groups first we remind some basic notions of group theory.

Recall that SL(2, C) is a topological group with the Euclidean metric topology. Hence, we obtain that PSL(2, C) = SL(2, C)\{±I}, where {±I} is a normal subgroup of SL(2, C), is a topological group with the quotient topology.

Suppose Γ is a group acting on a set X and let x ∈ X. We define the following:

(i) The stabilizer of x in Γ is the subgroup Γx = {γ ∈ Γ: γx = x}.
(ii) The Γ-orbit through x is the subset Γx = {γx: γ ∈ Γ} of X.
(iii) The orbit space of Γ on X is the set of all Γ-orbits X/Γ = {Γx: x ∈ X}.
(iv) A subset U of X is called Γ-invariant if γU = U for all γ ∈ Γ.

5.1 Discrete subgroups

A topological group is discrete if and only if all its elements are open.

There are other useful ways to determine discreteness. For example, if the identity element {1} is open in Γ then γ1 = γ is open for all γ ∈ Γ and so Γ is discrete.

A Kleinian group is a discrete subgroup of PSL (2, C).

The set of accumulation points of Γp in $\mathbb{C} \cup \infty$ is called the *limit set* of Γ, and usually denoted $\Lambda(\Gamma)$. The complement $\Omega(\Gamma) \coloneqq (\mathbb{C} \cup \infty) \setminus \big(\Lambda(\Gamma)\big)$ is called the domain of discontinuity or the ordinary set or the regular set.

Theorem 4.1. Let Γ be a Kleinian group. Then

(i) The limit set $\Lambda(\Gamma)$ is the smallest non-empty, closed and Γ-invariant set in $\mathbb{C} \cup \infty$.
(ii) The ordinary set $\Omega(\Gamma)$ is open, Γ-invariant in $\mathbb{C} \cup \infty$.
(iii) Let P denote the set of the fixed points of no elliptic elements of Γ, then $\Lambda(\Gamma) = \mathrm{P}$.
(iv) The limit set $\Lambda(\Gamma)$ is uncountable.
(v) If $\Omega(\Gamma) \neq \emptyset$, then $\Omega(\Gamma)$ is dense in $\mathbb{C} \cup \infty$ and $\Lambda(\Gamma)$ is nowhere dense in $\mathbb{C} \cup \infty$.

5.2 Apollonian group

Let $\mathcal{A}$ denote an Apollonian gasket. Then, the residual set of $\mathcal{A}$ is defined by

$$\Lambda(\mathcal{A}) \coloneqq \overline{\cup_{C \in A} C.}$$

Equivalently, if we take the complement in $C \cup \infty$. of the interiors of all circles in $\mathcal{A}$, we are left with the residual set $\Lambda(\mathcal{A})$.

Let us start with the Descartes configuration $\mathcal{D}_0 = \{C_1, C_2, C_3, C_4\}$ of an Apollonian gasket $\mathcal{A}$. Consider the group of Möbius transformations $S =< i_1, i_2, i_3, i_4 >$ acting on $\mathbb{C} \cup \infty$, where i_k is defined as an inversion with respect to the circle $C_{\tilde{k}}$ that passes through the tangency points of the circles C_l, for $l \neq k$. In other words, each inversion i_k fixes three of the initial circles in $\mathcal{D}_0$ (not pointwise) and acts reciprocally on the two tangent circles to those three. The set of the circles $C_{\tilde{k}}$ is called the dual Descartes configuration.

Notice that S leaves the gasket $\mathcal{A}$ invariant and there are four S-orbits of circles in $\mathcal{A}$. Hence, the group S generates the whole packing $\mathcal{A}$ through these inversions and the limit set $\Lambda(S)$ is equal to the residual set $\Lambda(\mathcal{A})$ of A. For this reason, the group S has been named the Apollonian group.

If we take the circle C_1 of the Descartes configuration $\mathcal{D}_0$ to be the unit circle, we can then easily calculate the analytic expression of the $i_k's$. The general expression of an inversion in a circle with radius r and center w is given by

$$i(z) = \frac{w\bar{z} + r^2 - w\bar{w}}{\bar{z} - \bar{w}}.$$

Then, the generators of the Apollonian group are

$$i_1(z) = \frac{\bar{z}}{-4i z + 1}, i_2 = \bar{z}$$

$$i_3(z) = \frac{(1+i)\bar{z} - 1}{\bar{z} - 1 + i}, i_4(z) = \frac{(-1+i)\bar{z} - 1}{\bar{z} + 1 + i}$$

The $i_k's$ are conformal, orientation-reversing maps. However, we can rewrite them as compositions of orientation-preserving maps, that is elements of PSL(2, C), and the complex conjugation map j. So,

$$i_k = \alpha_k \circ j,$$

where the $\alpha_k's$ are represented by the following matrices,

$$\alpha_1 = \begin{bmatrix} 1 & 0 \\ -4i & 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 1+i & -1 \\ 1 & -1+i \end{bmatrix}, \alpha_4 = \begin{bmatrix} -1+i & -1 \\ 1 & 1+i \end{bmatrix}.$$

Which are parabolic Möbius transformations.

6. Conclusions

The Apollonian Gasket is a fascinating geometric object that was constructed by iteration using Descartes's circle theorem. In this work was exposed the solution of Apollonius' problem from the perspective of the Greek school to the perspective of Felix Klein, that is, from the construction with ruler and compass to a topological categorization of the Apollonian Gasket.

We also know that by studying the recursion of the Apollonian problem we obtain the Apollonian Gasket which is a geometric object that can be constructed computationally; and in which, without a doubt, it is possible to appreciate the strength of the geometric study through complex functions.

The Apollonian sieve is a fractal structure of great interest to many mathematicians, who using Dynamical Systems, Number Theory, Measurement Theory and Dimension Theory tools know that:

(i) The Hausdorff dimension of Apollonian Gasket is bounded.
(ii) There exists a rational function whose Julia set is homeomorphic to the Sieve of Apollonius.

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